On robust observer design for a class of time-varying continuous-and discrete-time Persidskii systems

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***Abstract*— This paper considers the state estimation problem for a class of non-autonomous nonlinear systems. We propose conditions on the existence and stability of a nonlinear observer based on the invariant manifold approach in both continuous- and discrete-time scenarios. The requirements are formulated using Linear Matrix Equalities (LME) and Inequalities (LMI). We present two possible applications of the result, a reduced-order observer (*e.g.*, an observer for unmeasured states) and regression in linear and nonlinear, continuous- and discrete-time settings. With nonlinear regression being a sophisticated case, the parameter estimation problem for a particular output equation (when the fusion of linear and nonlinear sensors is weighted) is investigated. Two nonlinear examples demonstrating the efficiency of results are provided.**

I. Introduction

Designing observers is one of the central problems in modern control theory and dynamical systems analysis. Once presented for linear setting [20], the problem of observer design was vastly applied and studied for nonlinear cases (*e.g.*, [16] [7], [2], [27]) and became a popular research subject in the field. Reducing the order of the observer is a relevant sub-problem, the main idea being to separate the dynamics of measured states from unmeasured ones and disregard the estimate of known variables. This approach greatly simplifies the systems' analysis, modeling, and real-life applications. First stated in [20], such an idea presented for the linear case had many continuations in nonlinear analysis, some of them using linearization techniques [26], others based on solutions of partial differential equations (PDEs). Reduced-order observers were also studied for discrete-time systems both in linear [18] and nonlinear settings [4], [25].

The idea presented in this paper follows the so-called invariant manifold approach proposed in [14], [15], where it was specifically applied to reduced-order observers for a general class of nonlinear time-varying systems. It provides conditions for the existence of a solution, described by a nonlinear invertible function (chosen as a solution of the corresponding partial differential equation (PDE)), embedded into an invariant manifold containing both the observer and system dynamics. Recently, such a technique was used primarily in adaptive control and estimation and named the immersion and invariance approach [1]. The application of the invariant manifold method is non-trivial. It might lead to some difficulties related to the requirement of the existence of an invertible solution of a PDE, complicating its implementation (the choice of a solution is case-dependent).

In this note, we study a rather general class of nonlinear systems called Persidskii [23], including Lur'e systems, used in many areas, such as mechanical and electrical engineering, biology, genetics, and networks. Examples of such systems can be found, for instance, in neural networks [11], [13], electrical circuits [6], mechanical robotic systems [24], and bioreactors [8]. The chosen system class is of particular interest regarding the invariant manifold approach because a set of LMEs can replace the conventional PDE of the invariant manifold method, and a set of LMIs can provide conditions for the system's stability. As we show later in the paper, this procedure for the considered class of systems is more constructive in practice and easy to implement.

This paper also proves that our result can be applied to the reduced-order state observation problem and fits a conventional linear regression solution, a nonlinear one in some instances, displaying a broader field of implementation and problem-solving for a class of Persidskii systems. A preliminary version of this work can be found in [17]. Although this paper is based on this preliminary version, there are numerous advances in the present work as we consider a wider class of systems, with disturbances, and perform a deeper analysis of nonlinear regression.

The paper is structured as follows. The notation and preliminaries are presented in Section II, and the problem statement is in Section III. The main result on conditions of the observer existence and convergence is provided in Section IV for two cases: continuous- and discrete-time. Some applications, including reduced-order observers and parameter estimation in regression analysis (both in linear and nonlinear settings and in discrete time), are given in sections V and VI. Two nonlinear examples in Section VII show the efficiency of the presented estimators. A conclusion is given in Section VIII.

II. Preliminaries

A. Notation

- The sets of real and natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. Set $\mathbb{R}_+ := \{t \in \mathbb{R} \,|\, t \geq 0\}$. We denote by $\mathbb{R}^n$ the set of real vectors of dimension n and by $\mathbb{R}^{n\times m}$ the set of real matrices of dimension $n \times m$.
- For a Lebesgue measurable function $u : \mathbb{R}_+ \to \mathbb{R}^n$, define the norm $\|u\|_{[t_1,t_2)} = \text{ess sup}_{t\in[t_1,t_2)} \|u(t)\|$ for $[t_1,t_2) \subset \mathbb{R}_+$, where $\|\cdot\|$ is the Euclidian norm in $\mathbb{R}^n$. We denote by $\mathcal{L}_\infty^n$ the set of functions $u : \mathbb{R}_+ \to \mathbb{R}^n$ such that $\|u\|_\infty := \|u\|_{[0,+\infty)} < +\infty$.

[1] A. Khalin, R. Ushirobira and D. Efimov are with Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France {Anatolii.Khalin, Denis.Efimov, Rosane.Ushirobira}@inria.fr

- For a sequence $u = (u_k)_{k\in\mathbb{N}} \subset \mathbb{R}^n$, define the norm $|u|_\infty = \sup_{k\in\mathbb{N}} \|u_k\|$, and denote by L_∞^n the set of such sequences with finite norm $|u|_\infty < +\infty$.
- Let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimal and the maximal eigenvalues of a symmetric matrix A, respectively. Denote by I_n the $n\times n$ identity matrix.

We refer to [16] for definitions of input-to-state stability.

B. Input excitation

Definition 1. A function $\phi : \mathbb{R}_+ \to \mathbb{R}^n$ is *persistently exciting (or ϕ is PE)*, if there exist T, $\mu > 0$ such that for all $t \in \mathbb{R}_+$,

$$\int_t^{t+T} \phi(s)\phi^\top(s)ds > \mu I_n,$$

where $\phi^\top$ denotes the function $\mathbb{R}_+ \to \mathbb{R}^{1\times n}$, $t \mapsto \phi(t)^\top$.

C. Invariant manifold approach

Consider a time-varying nonlinear system in general form:

$$\begin{cases} \dot{y}(t) = f_1(y(t), x(t), t), \ \forall t \in \mathbb{R}_+ \\ \dot{x}(t) = f_2(y(t), x(t), t), \end{cases} \tag{1}$$

where $y(t) \in \mathbb{R}^m$ is the measured part of the state and $x(t) \in \mathbb{R}^n$ is the unmeasured part and the functions f_1 and f_2 are assumed to guarantee the forward completeness of (1).

Definition 2. The dynamical system

$$\dot{\xi}(t) = \alpha(y(t), \xi(t), t), \ \forall t \in \mathbb{R}_+ \tag{2}$$

where $\xi(t) \in \mathbb{R}^p$ $(p \geq n)$, is called an *observer* for the system (1) if there exist mappings

$$\beta : \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p \quad \text{and} \quad \phi_{y,t} : \mathbb{R}^n \to \mathbb{R}^p$$

with $\phi_{y,t}$ left-invertible and such that the manifold

$$\mathcal{M} = \{(y(t), x(t), \xi(t), t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} \mid \beta(y(t), \xi(t), t) = \phi_{y,t}(x(t))\} \tag{3}$$

has the following properties:

- all trajectories of the extended system (1), (2) starting on the manifold $\mathcal{M}$ remain there for all future times, *i.e.*, $\mathcal{M}$ is forward invariant;
- all trajectories of the extended system (1), (2) starting in a neighborhood of $\mathcal{M}$ asymptotically converge to $\mathcal{M}$.

To find the mappings β, α, and $\phi_{y,t}$, consider the estimation error:

$$z(t) = \beta(y(t), \xi(t), t) - \phi_{y,t}(x(t)),$$

the dynamics of which are given by

$$\dot{z} = \frac{\partial\beta}{\partial y} f_1 + \frac{\partial\beta}{\partial\xi}\alpha + \frac{\partial\beta}{\partial t} - \frac{\partial\phi_{y,t}}{\partial y} f_1 - \frac{\partial\phi_{y,t}}{\partial x} f_2 - \frac{\partial\phi_{y,t}}{\partial t}$$

and must admit the origin to be (locally) asymptotically stable. Existence conditions for these mappings formulated in terms of PDE, and a more detailed description of the approach can be found in [15].

III. Problem statement

To broaden our result's applicability, we consider in this paper a discrete-time setting and a continuous one.

A. Continuous-time

Consider a time-varying nonlinear system in the Persidskii form:

$$\begin{cases} \dot{x}(t) = A_0(t)x(t) + A_1(t)f(H(t)x(t)) + Q(t)u(t) + d(t), \\ y(t) = D_0(t)x(t) + D_1(t)f(H(t)x(t)) + v(t), \end{cases} \tag{4}$$

where $x : \mathbb{R}_+ \to \mathbb{R}^n$ is the state function and $y : \mathbb{R}_+ \to \mathbb{R}^p$ is the output function; $u : \mathbb{R}_+ \to \mathbb{R}^m$ is an essentially bounded external input function; $f : \mathbb{R}^r \to \mathbb{R}^\ell$ is a nonlinear continuous function, $A_0(t) \in \mathbb{R}^{n\times n}$, $A_1(t) \in \mathbb{R}^{n\times\ell}$, $H(t) \in \mathbb{R}^{r\times n}$, $Q(t) \in \mathbb{R}^{n\times m}$, $D_0(t) \in \mathbb{R}^{p\times n}$, $D_1(t) \in \mathbb{R}^{p\times\ell}$ are known time-varying matrices; $d \in \mathcal{L}_\infty^n$ and $v \in \mathcal{L}_\infty^q$ are unknown essentially bounded disturbances. We assume that f allows the forward existence and uniqueness of a solution of the system (4).

Let us build an observer for system (4) of the form:

$$\dot{\omega}(t) = S_0(t)\omega(t) + S_1(t)f(J(t)\omega(t)) + B(t)y(t) + O(t)u(t), \tag{5}$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}^q$ is the state function and $S_0(t) \in \mathbb{R}^{q\times q}$, $S_1(t) \in \mathbb{R}^{q\times\ell}$, $J(t) \in \mathbb{R}^{r\times q}$, $B(t) \in \mathbb{R}^{q\times p}$, $O(t) \in \mathbb{R}^{q\times p}$ are time-varying matrices to be chosen in a way allowing the estimation (of a part) of the state of (5) (see Definition 2).

B. Discrete-time

Now consider a time-varying nonlinear discrete-time system in the same form:

$$\begin{cases} x_{k+1} = A_{0,k}x_k + A_{1,k}f(H_kx_k) + Q_ku_k + d_k, \\ y_k = D_{0,k}x_k + D_{1,k}f(H_kx_k) + v_k, \end{cases} \tag{6}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$ are the state, bounded external input and the output vector at time instant $k \in \mathbb{N}$, respectively; $f : \mathbb{R}^r \to \mathbb{R}^\ell$ is a nonlinear function, $A_{0,k} \in \mathbb{R}^{n\times n}$, $A_{1,k} \in \mathbb{R}^{n\times\ell}$, $H_k \in \mathbb{R}^{r\times n}$, $Q_k \in \mathbb{R}^{n\times m}$, $D_{0,k} \in \mathbb{R}^{p\times n}$, $D_{1,k} \in \mathbb{R}^{p\times\ell}$ are known time-varying matrices at time instant k; $d \in L_\infty^n$ and $v \in L_\infty^q$ are unknown bounded disturbances.

Let us build an observer for system (6):

$$\omega_{k+1} = S_{0,k}\omega_k + S_{1,k}f(J_k\omega_k) + B_ky_k + O_ku_k, \tag{7}$$

where $\omega_k \in \mathbb{R}^q$ is the state-vector and $S_{0,k} \in \mathbb{R}^{q\times q}$, $S_{1,k} \in \mathbb{R}^{q\times\ell}$, $J_k \in \mathbb{R}^{r\times q}$, $B_k \in \mathbb{R}^{q\times p}$, $O_k \in \mathbb{R}^{q\times p}$ are time-varying matrices to be chosen.

This paper aims to establish conditions for using (5) and (7) as (reduced-order) observers for (4) and (6). Conditions for the existence of a static relationship between solutions of (4), (6), and (5), (7) have to be established (an analog of (3)). Also, convergence conditions must be set. Next, the goal is to apply the proposed observer to different estimation problems: reduced-order state observer design and parameter estimation in linear and nonlinear settings.

IV. Main result

To this end, the invariant manifold method [15] will be adapted to the presented problem statement to obtain a simple linear interconnection between the solutions of (4), (5), and (6), (7). However, instead of using the direct Lyapunov method for stability analysis of the error z, which would be analogous to the result in [15], we separately prove two properties of the manifold from Definition 2.

First, the existence conditions of a static relationship between solutions of the systems (4), (5), and systems (6), (7) should be established, meaning that the manifold connecting x, ω, and y is invariant. Next, to prove the attractiveness, we use the concept of globally convergent or incrementally input-to-state stable systems [22], *i.e.*, that if the observer state ω is initiated with different initial conditions (and with disturbances), then it asymptotically converges to the same solution (in its neighborhood). Thus, the found invariant solution is a steady-state one. This two-step approach allows us to obtain a constructive design for observer matrices.

A. Steady-state estimation

1) Continuous-time:

Proposition 1. *Assume that there exist time-varying matrices* $\Pi(t) \in \mathbb{R}^{q\times n}$, $\Upsilon(t) \in \mathbb{R}^{q\times p}$ *and also matrices* $S_0(t)$, $S_1(t)$, $B(t)$, $O(t)$, *that satisfy the following linear equalities* $\forall t \in \mathbb{R}_+$[1]*:*

$$\Upsilon D_1 = 0,\ J\left(\Pi + \Upsilon D_0\right) = H, \tag{8}$$

and

$$\begin{aligned} &S_0\left(\Pi + \Upsilon D_0\right) + \left(B - \dot{\Upsilon}\right) D_0 - \left(\Pi + \Upsilon D_0\right) A_0 - \dot{\Pi} - \Upsilon \dot{D}_0 = 0,\\ &BD_1 - \left(\Pi + \Upsilon D_0\right) A_1 + S_1 = 0,\\ &\left(\Pi + \Upsilon D_0\right) Q = O. \end{aligned} \tag{9}$$

Then, for $d = 0$ *and* $v = 0$*:*

$$\omega(t) = \Pi(t)x(t) + \Upsilon(t)y(t),\ t \in \mathbb{R}_+ \tag{10}$$

is a solution of the system (4), (5), for any $x(0) \in \mathbb{R}^q$ *and* $\omega(0) = \Pi(0)x(0) + \Upsilon(0)y(0)$.

Proof. Taking the derivative of (10) and using equations (4), (5), since $\Upsilon D_1 = 0$, we have:

$$\begin{aligned} &S_0(\Pi + \Upsilon D_0)x + S_1 f\left(J(\Pi x + \Upsilon y)\right) + B\left(D_0 x + D_1 f\left(Hx\right)\right)\\ &\quad + Ou = \left(\Pi + \Upsilon D_0\right)\left(A_0 x + A_1 f(Hx) + Qu\right) + \dot{\Pi}x + \dot{\Upsilon}y\\ &\qquad\qquad + \Upsilon\left(\dot{D}_0 x + \dot{D}_1 f(Hx)\right). \end{aligned}$$

Also $\dot{\Upsilon}D_1 = -\Upsilon\dot{D}_1$, so substituting (8) leads to the relation

$$\begin{aligned} &\left(S_0(\Pi + \Upsilon D_0) + (B - \dot{\Upsilon})D_0 - (\Pi + \Upsilon D_0)A_0 - \dot{\Pi} - \Upsilon\dot{D}_0\right)x\\ &= \left(-BD_1 + (\Pi + \Upsilon D_0)A_1 - S_1\right) f(Hx) + \left((\Pi + \Upsilon D_0)Q - O\right)u, \end{aligned}$$

which is satisfied thanks to (9) in a disturbance-free setting. □

[1] Throughout the paper, we simplify the notation for variables by writing, for instance, Π in place of $\Pi(t)$ when the time-dependency was once defined, unless the opposite is mentioned.

Remark 1. It is worth mentioning that the conditions of Proposition 1 restrict the class of systems to which the results can be applied. Note that there are additional free variables in (8) and (9), *i.e.*, matrices for the observer (5) to be set according to the design problem (see sections V-VII). We do not claim that a valid choice of matrices is always possible since it depends on the system's observability (on the problem's dimensions and the properties of matrices describing the dynamics of (4)). Nevertheless, other matrices in (8) and (9) can also be tuned if required (Υ and Π); hence the restrictions imposed in Propositions 1 are not significant.

2) Discrete-time:

Proposition 2. *Assume that for each* $k \in \mathbb{N}$ *there exist* $\Pi_k \in \mathbb{R}^{q\times n}$, $\Upsilon_k \in \mathbb{R}^{q\times p}$ *and also matrices* $S_{0,k}$, $S_{1,k}$, B_k, O_k *satisfying for each* $k \in \mathbb{N}$*:*

$$\Upsilon_k D_{1,k} = 0,\ J_k\left(\Pi_k + \Upsilon_k D_{0,k}\right) = H_k, \tag{11}$$

and

$$\begin{aligned} &S_{0,k}\left(\Pi_k + \Upsilon_k D_{0,k}\right) + B_k D_{0,k} - \left(\Pi_{k+1} + \Upsilon_{k+1} D_{0,k+1}\right) A_{0,k} = 0,\\ &B_k D_{1,k} - \left(\Pi_{k+1} + \Upsilon_{k+1} D_{0,k+1}\right) A_{1,k} + S_{1,k} = 0,\\ &\left(\Pi_{k+1} + \Upsilon_{k+1} D_{0,k+1}\right) Q_k = O_k. \end{aligned} \tag{12}$$

Then, for $d = 0$ *and* $v = 0$*:*

$$\omega_k = \Pi_k x_k + \Upsilon_k y_k,\ k \in \mathbb{N} \tag{13}$$

is a solution of the system (6), (7), for any $x_0 \in \mathbb{R}^q$ *and* $\omega_0 = \Pi_0 x_0 + \Upsilon_0 y_0$.

The proof of Proposition 2 is analogous to the continuous case, where instead of taking the derivative of (10) we take the next time instant for (13). Note that Remark 1 is applicable to Proposition 2 for (11) and (12) accordingly.

Observers (5), (7) of (4), (6) have the same shape of nonlinearity, so under suitable interconnections among the matrices in Propositions 1 and 2, the obtained relation between solutions is linear, reducing the complexity of analysis significantly and opening space for many applications.

B. Convergence to the invariant solution

In Propositions 1 and 2, only the existence of relations (10), (13) is proven. It should be established further whether this relation is attracting for (4), (6) and (5), (7) or not.

1) Continuous-time: Assume that the conditions of the Proposition 1 are verified:

Assumption 1. *There exist* Π *and* Υ *such that* $\Upsilon D_1 = 0$ *and the equalities (8) and (9) are satisfied for (4), (5).*

Next, consider the following dynamical system with a copy dynamics of (5):

$$\dot{\tilde{\omega}}(t) = S_0(t)\tilde{\omega}(t) + S_1(t) f(J(t)\tilde{\omega}(t)) + B(t)y(t) + O(t)u(t) + \delta(t),$$

where $\delta \in \mathcal{L}_\infty^q$ represents an exogenous perturbation (the influence of d and v). Let us introduce the error $e := \omega - \tilde{\omega}$ between two solutions of (5), initiated for different initial conditions, with the same inputs (y and u). Then we have the following dynamics:

$$\dot{e}(t) = S_0(t)e(t) + S_1(t)\left(f\left(J(t)\omega(t)\right) - f\left(J(t)\tilde{\omega}(t)\right)\right) - \delta(t). \tag{14}$$

Now we can state a theorem about the convergence and robust stability of the observer.

Theorem 1. *Let Assumption 1 be satisfied. Assume there exist $F(t) \in \mathbb{R}^{q\times k}$ and $W(t) = W(t)^\top \in \mathbb{R}^{q\times q}$ such that*

$$e^\top F\left(f(J\omega) - f(J\tilde{\omega})\right) \le e^\top We,$$

for all ω, $\tilde{\omega} \in \mathbb{R}^q$, and that there exist $P(t) = P(t)^\top \in \mathbb{R}^{q\times q}$, $\Xi(t) = \Xi(t)^\top \in \mathbb{R}^{q\times q}$ such that the inequalities

$$\alpha_1 I_n \le P,\ \alpha_1 I_q \le \Xi \quad P \le \alpha_2 I_n, \quad PS_1 = F,$$
$$\dot{P} + S_0^\top P + PS_0 + 2W + \Xi + \gamma P < 0$$

have a solution for some $0 < \alpha_1 < \alpha_2 < +\infty$ and $\gamma > 0$. Then, the system (14) is input-to-state stable (ISS) and (5) is globally convergent.

Proof. Assumption 1 ensures the existence of an estimate (10). So, select a Lyapunov function $V(e) = e^\top Pe$, with P given by the theorem conditions (this Lyapunov function is positive definite and radially unbounded in e due to the introduced restriction). Its derivative for (14) takes the form:

$$\dot{V} = e^\top(S_0^\top P + PS_0 + \dot{P})e + 2e^\top PS_1\left(f(J\omega) - f(J\tilde{\omega})\right) - 2e^\top P\delta$$
$$\le e^\top\left(S_0^\top P + PS_0 + \dot{P} + 2W + \Xi + \gamma P\right)e - e^\top \Xi e + \gamma^{-1}\delta^\top P\delta,$$

where $F = PS_1$ and W, γ are given in the formulation of the theorem. According to the imposed conditions, the above expression is non-positive and:

$$\|e(t)\| \le \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\left(e^{-0.5\frac{\lambda_{\min}(\Xi)}{\lambda_{\max}(P)}t}\|e(0)\| + \gamma^{-0.5}\|\delta\|\right),\ \forall t \ge 0.$$

□

The conditions presented in Theorem 1 are given for an illustration, and any other conditions for convergence to zero and robust stability of e in (14) as in [21] can be used.

2) Discrete-time: Assume that the conditions of the Proposition 2 are verified:

Assumption 2. *There exist Π_k and Υ_k such that $\Upsilon_k D_{1,k} = 0$ and the equalities (11) and (12) are satisfied for (6), (7).*

Next, consider the following dynamical system with a copy dynamics of (7):

$$\tilde{\omega}_{k+1} \quad = \quad S_{0,k}\tilde{\omega}_k + S_{1,k}f(J_k\tilde{\omega}_k) + B_k y_k + O_k u_k + \delta_k,$$

where $\delta \in L_\infty^q$ is an auxiliary bounded disturbance as before. Let us introduce the error $e_k := \omega_k - \tilde{\omega}_k$ between two solutions of (7), initiated for different initial conditions, with the same inputs (y_k and u_k). Then we have the following system:

$$e_{k+1} = S_{0,k}e_k + S_{1,k}\left(f(J_k\omega_k) - f(J_k\tilde{\omega}_k)\right) - \delta_k. \tag{15}$$

Now we can state a theorem about the convergence and robust stability of the observer.

Theorem 2. *Let Assumption 2 be satisfied. Assume there exists $\gamma > 0$ such that*

$$|f(J_k\omega_k) - f(J_k\tilde{\omega}_k)| \le \gamma|e_k|$$

for all ω_k, $\tilde{\omega}_k \in \mathbb{R}^q$, and that there exist $P_k = P_k^\top \in \mathbb{R}^{q\times q}$, $\Xi = \Xi^\top \in \mathbb{R}^{q\times q}$ and $\Gamma = \Gamma^\top \in \mathbb{R}^q$ such that the inequalities

$$\begin{pmatrix} \Sigma_k & S_{0,k}^\top P_{k+1}S_{1,k} & -S_{0,k}^\top P_{k+1} \\ S_{1,k}^\top P_{k+1}S_{0,k} & S_{1,k}^\top P_{k+1}S_{1,k} - \alpha I & -S_{1,k}^\top P_{k+1} \\ -P_{k+1}S_{0,k} & -P_{k+1}S_{1,k} & P_{k+1} - \Gamma \end{pmatrix} \le 0,$$
$$\alpha_1 I_n \le P_k \le \alpha_2 I_n,\ \alpha_1 I_q \le \Xi$$

where $\Sigma_k = S_{0,k}^\top P_{k+1}S_{0,k} - P_k + \Xi + \alpha\gamma^2 I$, have a solution for some $0 < \alpha_1 < \alpha_2 < +\infty$ and $\alpha > 0$. Then the system (15) is ISS and (7) is globally convergent.

Proof. Assumption 2 ensures the existence of an estimate (13), so let us select a Lyapunov function $V_k = e_k^\top P_k e_k$, where P_k is given in the conditions of the theorem. Consider the difference $V_{k+1} - V_k$ and denote $\Delta f_k := f(J_k\omega_k) - f(J_k\tilde{\omega}_k)$ then using the bound defined in the theorem, we have

$$V_{k+1} - V_k \le \begin{pmatrix} e_k \\ \Delta f_k \\ \delta_k \end{pmatrix}^\top \Phi_k \begin{pmatrix} e_k \\ \Delta f_k \\ \delta_k \end{pmatrix} - e_k^\top \Xi e_k + \delta_k^\top \Gamma \delta_k.$$

where Φ_k denotes the matrix in the theorem's statement. According to the imposed conditions, $\Phi_k \le 0$ and the error e_k is ISS for the input δ_k. □

For brevity, the formulations of the results in this section deal with global stability properties based on the global characteristics of the nonlinearities. It is possible to get local stability results by restricting the domain of validity of the constraints for f, as it will be illustrated in Section VII.

V. Reduced-order observer design

Let us demonstrate how the generic results presented in the previous section can be used to estimate the system's unobserved states.

A. Linear reduced-order observer

The first application of the result presented in the previous section is a reduced-order observer for the linear case, where $S_1 = 0$, $D_1 = 0$, and all other matrices are known and constant in (4) (the time-invariance has been imposed to simplify the presentation and comparison). Then we have an ordinary LTI system of the form:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + Qu(t) + d(t),\ t \in \mathbb{R}_+, \\ y(t) = D_0 x(t) + v(t). \end{cases} \tag{16}$$

As in the classical problem of reduced-order observer design, we can partition our state $x(t) \in \mathbb{R}^n$ with variables $y(t) \in \mathbb{R}^p$ representing a set of directly measured state variables and $w(t) \in \mathbb{R}^{n-p}$ as a set of unmeasured states:

$$\begin{pmatrix} y(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} D_0 \\ \Pi \end{pmatrix} x(t) + \begin{pmatrix} v(t) \\ 0 \end{pmatrix}. \tag{17}$$

In practice, v is often considered a bounded measurement noise. The task is to build an observer, effectively reducing the dynamic order of (5) from n to $n-p$. Therefore, consider the following LTI system (the respective presentation of (5)):

$$\dot{\omega}(t) = S_0\omega(t) + By(t) + Ou(t), \tag{18}$$

where $\omega(t) \in \mathbb{R}^{n-p}$ is the observer state, and $S_0 \in \mathbb{R}^{(n-p)\times(n-p)}$, $B \in \mathbb{R}^{(n-p)\times p}$, $O \in \mathbb{R}^{(n-p)\times m}$ are constant matrices to be determined.

Then, Proposition 1 can be applied for the considered case. If the following matrix equalities are verified:

$$\begin{aligned} &S_0(\Pi + \Upsilon D_0) + BD_0 - (\Pi + \Upsilon D_0)\, A_0 = 0, \\ &(\Pi + \Upsilon D_0)\, Q = O, \end{aligned} \tag{19}$$

then there exists a solution

$$\omega(t) = \Pi x(t) + \Upsilon y(t), \ \forall t \in \mathbb{R}_+ \tag{20}$$

for any $x(0) \in \mathbb{R}^q$ and $\omega(0) = \Pi x(0) + \Upsilon y(0)$, connecting (16) and (18). Applying Assumption 1 to (20), (16), and (18), we can use Theorem 1 (we ask for the existence of a positive definite symmetric matrix $P \in \mathbb{R}^{p\times p}$ such that $S_0^\top P + PS_0 < 0$) to show that (18) is globally asymptotically stable (the estimation error is input-to-state stable). Furthermore, a global asymptotic observer for w, makes it a reduced-order observer for (16). This classical result is well-known and was first presented in [20]. The purpose of considering such a case here is to demonstrate that the traditional result for linear systems can be obtained through the previous idea.

B. Nonlinear reduced-order observer

Consider (4) and (5) with constant matrices for simplicity, and let $D_1 = 0$. We have a nonlinear system:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + A_1 f(Hx(t)) + Qu(t) + d(t), \ t \in \mathbb{R}_+, \\ y(t) = D_0 x(t) + v(t), \end{cases} \tag{21}$$

where $x(t) \in \mathbb{R}^n$ is the full state, $y(t) \in \mathbb{R}^p$ is the measured part of the state and A_0, A_1, Q, H are matrices of corresponding dimensions, $d \in \mathcal{L}_\infty^n$ is an external disturbance or a modeling error and $v \in \mathcal{L}_\infty^p$ is a measurement noise. Our goal is to design a reduced-order observer for (21) of a smaller dimension $q = n - p$:

$$\dot{\omega}(t) = S_0\omega(t) + S_1 f(J\omega(t)) + By(t) + Ou(t), \tag{22}$$

where $\omega(t) \in \mathbb{R}^q$, and constant matrices are of corresponding dimensions. All of them have to be defined using the presented method. Proposition 1 gives us the existence conditions of a solution in a disturbance-free scenario ($d = 0$, $v = 0$ and Π is obtained from the representation (17)):

$$\omega(t) = \Pi x(t) + \Upsilon y(t) = (\Pi + \Upsilon D_0)x(t) = Zx(t), \tag{23}$$

where $Z \in \mathbb{R}^{q\times n}$ is a matrix connecting solutions of the initial system (21) and the observer (22). Then, (8) and (9) in this case are as follows:

$$\begin{aligned} &JZ = H, \\ &S_0 Z = ZA_0 - BD_0, \\ &S_1 = ZA_1, \\ &O = ZQ. \end{aligned} \tag{24}$$

The equations above can be solved with respect to S_0, S_1, B, O and J, knowing Z. Theorem 1 can be directly applied with a constant matrix P to prove the convergence of the observer (22) to a hyperplane given by (23). Since the order of (22) is lower than the one of (21), we can say that (22) is an asymptotical reduced-order observer.

Remark 2. The same result can be derived for time-varying matrices. The difference will lay only in matrix equations from Proposition 1 and in the form of $S_0(t)$ and $S_1(t)$.

C. Discrete-time reduced-order observer

Consider (6) and (7) and let $D_{1,k} = 0$, $k \in \mathbb{N}$. We have a nonlinear discrete-time system:

$$\begin{cases} x_{k+1} = A_{0,k}x_k + A_{1,k} f(H_k x_k) + Q_k u_k + d_k, \ k \in \mathbb{N}, \\ y_k = D_{0,k}x_k + v_k, \end{cases} \tag{25}$$

where $x_k \in \mathbb{R}^n$ is the full state, $y_k \in \mathbb{R}^p$ is the measured part of the state and $A_{0,k}$, $A_{1,k}$, Q_k, H_k are matrices of corresponding dimensions at time k, d_k is an external disturbance and v_k is a measurement noise. Then we build an observer again:

$$\omega_{k+1} = S_{0,k}\omega_k + S_{1,k} f(J_k\omega_k) + B_k y_k + O_k u_k, \tag{26}$$

where $\omega_k \in \mathbb{R}^q$, and all matrices are of corresponding dimensions. Proposition 2 gives us the existence conditions for a solution in a disturbance-free scenario and Π_k are as follows:

$$\omega_k = \Pi_k x_k + \Upsilon_k y_k = (\Pi_k + \Upsilon_k D_{0,k})x_k = G_k x_k, \tag{27}$$

where $G_k \in \mathbb{R}^{q\times n}$ is a matrix at time instant k, connecting solutions of the initial system (25) and the observer (26). Then, (11) and (12) are as follows:

$$\begin{aligned} &S_{0,k}G_k + B_k D_{0,k} - G_{k+1}A_{0,k} = 0, \\ &B_k D_{1,k} - G_{k+1}A_{1,k} + S_{1,k} = 0, \\ &G_{k+1}Q_k = O_k, \\ &J_k G_k = H_k. \end{aligned} \tag{28}$$

The equations above can be solved to determine $S_{0,k}$, $S_{1,k}$, B_k, O_k, and J_k, knowing G_k. Theorem 2 can be directly applied with a matrix P_k to prove the convergence of (26).

The results on reduced-order observer design for nonlinear systems in [26] and [25] are local and based on linearization analysis, while the general results presented in [15] cover the considered case, but they are less constructive due to the requirement of the existence of a solution for the PDE.

VI. Regression

Let us consider another possible application of the result presented in Section IV.

A. Linear regression

A well-known and common method in practical estimation problems is the linear parameter regression. The application of our result is less intuitive in this setting than previously. However, it fits the problem statement rather well. To demonstrate it, let us consider a simple linear regression equation:

$$\dot{x}(t) = d(t), \ t \in \mathbb{R}_+ \tag{29}$$

$$y(t) = D_0(t)x(t) + v(t), \tag{30}$$

where $y(t) \in \mathbb{R}^p$ is a measured output, $D_0(t) \in \mathbb{R}^{p\times n}$ is a regression matrix, $x(t) \in \mathbb{R}^n$ is the vector of unknown parameters to be estimated, which in our case is the state-vector and $d(t) \in \mathbb{R}^n$ is a bounded disturbance signal. Having nonzero disturbance means that the parameter vector x may not be necessarily constant but can be affected by external factors and vary in time in a certain region. In the case of linear regression, the disturbance v can represent an unmodeled observational or experimental error. To apply Proposition 1, we consider $A_0 \equiv A_1 \equiv D_1 \equiv Q \equiv 0$ in (4) and we represent the parameter observer in the required form (5) with $S_1 \equiv O \equiv 0$:

$$\dot{\omega}(t) = S_0(t)\omega(t) + B(t)y(t), \tag{31}$$

Then, we can choose $\Pi \equiv I_n$ as the constant identity matrix, $\Upsilon \equiv 0$, leading to:

$$\omega(t) = x(t), \tag{32}$$

meaning that (31) is an observer for x. Applying Proposition 1, we have the equality:

$$S_0 = -BD_0. \tag{33}$$

Since we need to choose the matrices S_0 and B, let us assign $B = \Gamma D_0^\top$, where Γ is a nonsingular matrix, and can substantiate $S_0 = -\Gamma D_0^\top D_0$. Substituting in (31), it results in a conventional gradient estimator:

$$\dot{\omega}(t) = \Gamma D_0^\top(t)\left(y(t) - D_0(t)\omega(t)\right). \tag{34}$$

For the asymptotic convergence of this observer, we need a standard additional assumption [3]:

Assumption 3. *The matrix $D_0(t)$ is PE, for $\forall t \in \mathbb{R}_+$.*

Remark 3. There exist techniques to relax the PE condition using preprocessing algorithms, such as the Dynamic Regressor Extension and Mixing (DREM) method (see, for example, [9]). To demonstrate our approach in this paper, we prefer to keep Assumption 3 for the sake of generality.

If Assumption 3 is satisfied, then applying Theorem 1 to (32), (30), and (31), we obtain the convergence of the estimator to the state x (under Assumption 1, it is expected that there is a matrix P satisfying the conditions of the theorem). This result is well known for parameter estimation in linear regression (see, for instance, [19]). Despite the result not being novel, this application demonstrates the broad applicability of the approach presented in Section IV to estimation problems.

B. Nonlinear regression

The estimation problem for nonlinear regression is more complex and might require some additional conditions to guarantee the observer's convergence. Therefore, let us consider a particular case:

$$\dot{x}(t) = d(t), \; t \in \mathbb{R}_+ \tag{35}$$

$$y(t) = (1 - D_1(t))J(t)x(t) + D_1(t)f(J(t)x(t)), \tag{36}$$

where $D_0(t) = (1 - D_1(t))J(t)$ and $0 \le D_1(t) \le 1$, for all $t \ge 0$, $A_0 \equiv A_1 \equiv Q \equiv 0$, $y(t) \in \mathbb{R}^p$ is measured output vector, $D_1(t) \in \mathbb{R}^{p\times\ell}$, $J(t) \in \mathbb{R}^{r\times n}$ are regression matrices, $x(t) \in \mathbb{R}^n$ is a vector of unknown parameters and $d(t) \in \mathbb{R}^n$ is a bounded disturbance signal.

Remark 4. A form of regression equation (36) might be found in models where the output is a fusion of data from multiple sensors (*i.e.*, linear and nonlinear in parameters) of different nature, whose combination is used to improve the estimation [10]. For instance, the possible failure [5] or missing data [12] from some sensors is often considered. In any case, D_1 is considered a signal that dynamically weights the data from linear and nonlinear sensors.

We set $\Pi \equiv I_n$ and $\Upsilon \equiv 0$, so we are looking for a solution in the form $\omega \equiv x$. Using Proposition 1, we obtain the following equalities:

$$J = H,$$
$$S_0 + B(1 - D_1)J = 0,$$
$$S_1 + BD_1 = 0.$$

Let $B = \gamma J^\top$ for some $\gamma > 0$, then, we have the corresponding observer:

$$\dot{\omega}(t) = -\gamma(1 - D_1(t))J(t)^\top J(t)\omega(t) - \gamma D_1(t)J(t)^\top f(J(t)\omega(t)) + \gamma J^\top y(t), \tag{37}$$

Consider a copy dynamics of (37):

$$\dot{\tilde{\omega}}(t) = -\gamma(1 - D_1(t))J(t)^\top J(t)\tilde{\omega}(t) - \gamma D_1(t)J(t)^\top f(J(t)\tilde{\omega}(t)) + B(t)y(t) + \delta(t),$$

then we have error dynamics for $e(t) = \omega(t) - \tilde{\omega}(t)$:

$$\dot{e}(t) = -\gamma(1 - D_1(t))J(t)^\top J(t)e(t) - \gamma D_1(t)J(t)^\top\left(f(J(t)\omega(t)) - f(J(t)\tilde{\omega}(t))\right) - \delta(t). \tag{38}$$

Remark 5. If $D_1(t) = 0, \forall t \in \mathbb{R}_+$, then (36) takes the form of linear regression equation (30), where $J \equiv D_0$.

Theorem 1 can be used for stability analysis of (38), but to get more efficient stability conditions, let us consider its modification. For this purpose, we need an auxiliary hypothesis:

Assumption 4. *The function J is PE.*

Under this condition, consider a time-dependent matrix

$$P(t) = \int_t^\infty \Phi(s,t)^\top \Phi(s,t)\,ds,$$

where

$$\dot{\Phi}(t,t_0) = -\gamma J^\top(t)J(t)\Phi(t,t_0) \quad \Phi(t_0,t_0) = I.$$

then the following equality is satisfied

$$\dot{P}(t) - \gamma P(t)J(t)^\top J(t) - \gamma J^\top(t)J(t)P(t) = -I_n. \tag{39}$$

Theorem 3. *Let Assumptions 4 and 1 be satisfied. Assume that*

$$-\gamma D_1 e^\top P J^\top\left(f(J\omega) - f(J\tilde{\omega}) - Je\right) < \frac{e^\top e}{2},$$

for all ω, $\tilde{\omega} \in \mathbb{R}^q$ and $e = \omega - \tilde{\omega}$, with P defined as above. Then, the system (38) is ISS and (37) is globally convergent.

Proof. Assumption 1 ensures the existence of an invariant solution, so select a Lyapunov function candidate $V(e) = e^\top Pe$. Then the derivative of V along the trajectories of (38) takes the form:

$$\dot{V} = e^\top \dot{P} e + \left(-\gamma J^\top J e - \gamma D_1 J^\top \left(f(J\omega) - f(J\tilde{\omega}) - Je\right)\right)^\top P(t) e$$
$$+ e^\top P \left(-\gamma J^\top J(t) e - \gamma D_1 J^\top \left(f(J\omega) - f(J\tilde{\omega}) - Je\right)\right).$$

Substituting the expression of the derivative of P we have:

$$\dot{V} = -e^\top e - 2\gamma D_1(t) e^\top P J^\top \left(f(J\omega) - f(J\tilde{\omega}) - Je\right),$$

Using the inequality of the theorem, we obtain $\dot{V} < 0$ and asymptotic stability of the error e. □

The current section demonstrates an advantage of the generality of the approach considered in Section IV, which allows the investigation of a rather wide range of linear and nonlinear estimation problems.

VII. Examples

Example 1. Let us consider a two mass-spring system with nonlinear stiffness affected by the white Gaussian noise v and disturbance d, the dynamics of which can be expressed as follows:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -k_1(x_1 - x_3) - k_2(x_1 - x_3)^3 - a_1(x_2 - x_4) + d, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = k_1(x_1 - x_3) + k_2(x_1 - x_3)^3 + a_1(x_2 - x_4) - k_3 x_3 - a_2 x_4 + u, \\ y_1 = x_1 + v, \\ y_2 = x_2 + v \end{cases} \tag{40}$$

with $u(t) = \sin(t)$ and $d(t) = 0.2\sin(10t)$, $t \in \mathbb{R}_+$. Presenting the system in the form (4), we have:

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k_1 & -a_1 & k_1 & a_1 \\ 0 & 0 & 0 & 1 \\ k_1 & a_1 & -k_1 - k_3 & a_1 - a_2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 \\ -k_2 \\ 0 \\ k_2 \end{pmatrix},$$

$$f(\mu) = \mu^3 \ \ (\mu \in \mathbb{R}), \quad Q = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^\top,$$

$$D_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix}, \quad D_1 = 0,$$

where the states x_1 and x_2 correspond to the position and the velocity of the first mass, assumed to be measured. The states x_3 and x_4 represent the second mass's position and velocity, which we need to estimate. The signal u is the periodic force applied to the second mass to excite the system. The task is to build an observer of the form (22) for the states x_3 and x_4 using Proposition 1 (or (26) and Proposition 2). We have the following LMEs:

$$\begin{aligned} J(\Pi + \Upsilon D_0) &= H, \\ S_0(\Pi + \Upsilon D_0) &= (\Pi + \Upsilon D_0)A_0 - BD_0, \\ S_1 &= (\Pi + \Upsilon D_0)A_1, \\ O &= (\Pi + \Upsilon D_0)Q. \end{aligned} \tag{41}$$

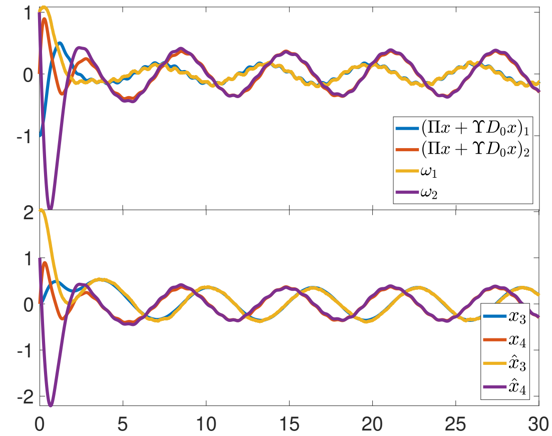

Fig. 1. Example 1. Reduced-order observer for a mass-spring system.

We can assign Π and Υ having full row rank (the matrix Π should have a left inverse with respect to the unmeasured state components of (40), while Υ describes the utilization of the output variables) as, for example:

$$\Pi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1, \end{pmatrix} \quad \Upsilon = \begin{pmatrix} \upsilon_1 & 0 \\ 0 & \upsilon_2 \end{pmatrix}.$$

Therefore, we have 16 equations in total, with 16 unknowns. Solving the equations for chosen simulation values of the coefficients ($k_1 = 3$, $k_2 = 3$, $k_3 = 0.6$, $a_1 = 0.6$, $a_2 = 2$), we obtain:

$$S_0 = \begin{pmatrix} 0 & 1 \\ -3.6 & -2.6 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad J = \begin{pmatrix} -1 & 0 \end{pmatrix},$$

$$O = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ -0.6 & 0.6 \end{pmatrix}, \quad \Upsilon = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, we have the dynamics of ω, which must satisfy the LMIs in Theorem 1. From the given dynamics, we have the condition:

$$e^\top F\left(\tilde{\omega}_1^3 - \omega_1^3\right) = (F_1 e_1 + F_2 e_2)\left(-e_1(\tilde{\omega}_1^2 + \omega_1\tilde{\omega}_1 + \omega_1^2)\right) \le e^\top We,$$

which has to be satisfied locally in $\tilde{\omega}$, $\omega \in \mathbb{R}^2$ by imposing the admissible upper limits on these variables. Assume that $\tilde{\omega}_1^2 + \omega_1^2 \le \eta$ for some $\eta > 0$, then

$$W = 0.5\eta \begin{pmatrix} -F_1 + 1.5F_2 & 0 \\ 0 & 1.5F_2 \end{pmatrix}.$$

Consider, for example the following matrices:

$$F = \begin{pmatrix} 0.25 \\ 0 \end{pmatrix}, \quad W = \begin{pmatrix} -0.1875 & 0 \\ 0 & 0 \end{pmatrix}.$$

For $\gamma = 0.5$, $\eta = 1.5$ the solution of LMIs given by the solver:

$$P = \begin{pmatrix} 0.267 & 0.0547 \\ 0.0547 & 0.031 \end{pmatrix}, \quad \Xi = \begin{pmatrix} 0.384 & -0.006 \\ -0.006 & 0.027 \end{pmatrix}.$$

Finally, we build an observer for states x_3 and x_4 from the expression $\omega = \Pi x + \Upsilon D_0 x$:

$$\hat{x}_3 = \frac{\omega_1 - \upsilon_1 x_1}{p_1} = \omega_1 + x_1, \quad \hat{x}_4 = \frac{\omega_2 - \upsilon_4 x_2}{p_4} = \omega_2.$$

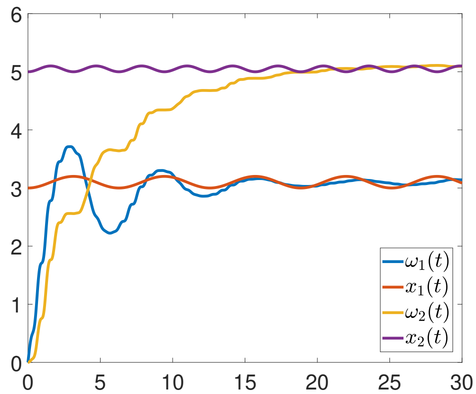

Fig. 2. Example 2. Nonlinear regression for a system given in a form (36).

Fig. 1 demonstrates the convergence of the observer for unmeasured states, successfully reducing the dimension of the observation problem from 4 to 2.

Example 2. Consider the nonlinear regression equation (36) with $d(t) = 0.1\left(\sin(t) \quad \sin(2t)\right)^\top$, error $v \equiv 0$ and

$$J(t) = \left(1 \quad \sin(t)\right), \quad D_1(t) = \sin^2(5t), \quad f(t) = \arctan(t)$$

for all $t \in \mathbb{R}_+$, and constant parameters $x(0) = \left(3 \quad 5\right)^\top$ to estimate. Then we build an observer of the form (37), with $\gamma = 0.75$, and the conditions of the Theorem 3 are satisfied. Fig. 2 demonstrates the convergence of the estimates ω to the time-varying parameter x.

VIII. Conclusion

The results presented in this paper significantly simplify the design of reduced-order observers for a particular class of systems, avoiding solutions of PDEs that arise in the conventional invariant manifold methodology. The resulting solution is explicit and more constructive than existing results on nonlinear observer design. We have shown two possible applications of our approach in discrete- and continuous-time cases: nonlinear reduced-order observer and nonlinear regression. Two examples, one representing a mechanical system and an academic case for nonlinear regression, demonstrated the applicability of our results in this work.

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