On observer design for a class of time-varying Persidskii systems based on the invariant manifold approach

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***Abstract*— In this work, we consider the state estimation problem for a class of non-autonomous Persidskii systems. This paper presents conditions on the existence and stability of a nonlinear observer based on the invariant manifold approach. The conditions are formulated using Linear Matrix Equalities (LME) and Inequalities (LMI). Two interesting applications of the result are presented: a reduced-order observer (*e.g.*, an observer for unmeasured states) and regression, both in linear and nonlinear settings. An example to demonstrate the efficiency of results is provided.**

I. Introduction

Designing observers is one of the central problems in modern control theory and in dynamical systems analysis. Once presented for linear cases [14], the problem of observer design was vastly applied and studied for nonlinear cases (*e.g.*, [12] [8], [4], [18]) and became a popular research subject in the field. The issue of order reduction for the observer is a relevant sub-problem. The main idea is to separate the dynamics of measured states from unmeasured ones and not estimate unnecessary variables (known). This procedure greatly simplifies the analysis, models, and real-life applications. First stated in [14], such a problem for the linear case had many continuations in nonlinear analysis, starting from using linearization techniques [17] and continuing with observer design based on solutions of partial differential equations (PDEs).

The idea presented in this paper follows the so-called invariant manifold approach presented in [10]. It was used specifically to provide conditions on the existence of a solution for the corresponding reduced-order observer for a general class of nonlinear time-varying systems. The solution lies in an invariant manifold described by a nonlinear invertible function (chosen as a solution of the corresponding PDE). Later, this method was extended in [11]. Recently, such a technique was used primarily in adaptive control and estimation, where it is named the inversion and invariance approach [2]. Applying the invariant manifold method is non-trivial. It might lead to many difficulties (related to the requirement of the existence of an invertible solution of a PDE), which makes its implementation complicated (the choice of a solution may be case-dependent).

In this note, we study a rather general class of nonlinear systems in the Persidskii form [15], including Lur'e systems, widely used in mechanical and electrical engineering modeling. Persidskii systems are common in practice and can be found, for example, in neural networks [7], [9], electrical circuits [6], mechanical robotic systems [16], and bioreactors [3]. The advantage of such a choice is that LMEs can replace the conventional PDE of the invariant manifold method, and LMIs provide system stability. This procedure emerges to be more constructive in practice and easy to implement, as we will demonstrate.

In addition, this paper shows that the presented result can be applied not only to the reduced-order state observation problem but also fits a conventional linear regression solution and a nonlinear one, which shows a broader field of implementation and problem solving for Persidskii systems.

The paper is structured as follows. Notation and Problem statement are presented in Section II. The main result on conditions of the observer existence and convergence is provided in Section III. Some applications, including reduced-order observers and parameter estimation in regression analysis (both in linear and nonlinear settings), are given in Section IV. A nonlinear example in section VI demonstrates the efficiency of the presented work. A conclusion with some remarks is given in VII.

II. Preliminaries

A. Notation

- The set of real numbers is denoted by $\mathbb{R}$, and we write $\mathbb{R}_+ := \{t \in \mathbb{R} \,|\, t \geq 0\}$. The spaces of real vectors of dimension n and real matrices of dimension $n \times m$ are denoted by $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, respectively.
- For a Lebesgue measurable function $u : \mathbb{R}_+ \to \mathbb{R}^n$, define the norm $\|u\|_{[t_1,t_2)} = \text{ess}\sup_{t\in[t_1,t_2)} \|u(t)\|$ for $[t_1, t_2) \subset \mathbb{R}_+$, where $\|\cdot\|$ refers to the Euclidian norm in $\mathbb{R}^n$. We denote by $\mathcal{L}_\infty^n$ the set of functions $u : \mathbb{R}_+ \to \mathbb{R}^n$ such that $\|u\|_\infty = \|u\|_{[0,+\infty)} < +\infty$.
- Let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimal and the maximal eigenvalues of a symmetric matrix A, respectively. Denote by I_n the $n \times n$ identity matrix.

B. Input excitation

Definition 1. A function $\phi : \mathbb{R}_+ \to \mathbb{R}^n$ is *persistently excited (or ϕ is PE)*, if there exist $T, \mu > 0$ such that for all $t \in \mathbb{R}_+$,

$$\int_t^{t+T} \phi(s)\phi^\top(s)ds > \mu I_n,$$

where $\phi^\top$ denotes the function $\mathbb{R}_+ \to \mathbb{R}^{1\times n}$, $t \mapsto \phi(t)^\top$.

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Definition 2. Two PE functions $\phi:\mathbb{R}_+\to\mathbb{R}^n$, $\psi:\mathbb{R}_+\to\mathbb{R}^m$ are *linearly independent*, if there exists $T>0$ such that for all $t\in\mathbb{R}_+$, the following matrix functions:

$$\left(\int_t^{t+T}\phi(s)\phi(s)^\top ds\right)^{-1}\int_t^{t+T}\phi(s)\psi(s)^\top ds \text{ and}$$
$$\left(\int_t^{t+T}\psi(s)\psi(s)^\top ds\right)^{-1}\int_t^{t+T}\psi(s)\phi^\top(s)ds$$

are not constant.

C. Problem statement

Consider a time-varying nonlinear system in Persidskii form:

$$\begin{cases}\dot{x}(t)=A_0(t)x(t)+A_1(t)f\left(H(t)x(t)\right)+Q(t)u(t),\ t\in\mathbb{R}_+,\\ y(t)=D_0(t)x(t)+D_1(t)f\left(H(t)x(t)\right),\end{cases}\tag{1}$$

where $x:\mathbb{R}_+\to\mathbb{R}^n$ is the state function and $y:\mathbb{R}_+\to\mathbb{R}^p$ is the output function; $u:\mathbb{R}_+\to\mathbb{R}^m$ is an essentially bounded external input function; $f:\mathbb{R}^r\to\mathbb{R}^k$ is a nonlinear continuous function, $A_0(t)\in\mathbb{R}^{n\times n}$, $A_1(t)\in\mathbb{R}^{n\times k}$, $H(t)\in\mathbb{R}^{r\times n}$, $Q(t)\in\mathbb{R}^{n\times m}$, $D_0(t)\in\mathbb{R}^{p\times n}$, $D_1(t)\in\mathbb{R}^{p\times k}$ are known time-varying matrices. We assume that the function f allows the forward existence and uniqueness of a solution of the system (1).

Let us build an observer for system (1) of a form:

$$\dot{\omega}(t)=S_0(t)\omega(t)+S_1(t)f\left(J(t)\omega(t)\right)+B(t)y(t)+O(t)u(t),\tag{2}$$

where $\omega:\mathbb{R}_+\to\mathbb{R}^q$ is the state function and $S_0(t)\in\mathbb{R}^{q\times q}$, $S_1(t)\in\mathbb{R}^{q\times k}$, $J(t)\in\mathbb{R}^{r\times q}$, $B(t)\in\mathbb{R}^{q\times p}$, $O(t)\in\mathbb{R}^{q\times p}$ are time-varying matrices to be chosen.

This paper aims to establish conditions for using (2) as a (reduced-order) observer for (1). Conditions for the existence of a static relationship between solutions of (1) and (2) have to be established (what is estimated). Also, convergence conditions must be set. Next, the goal is to apply the proposed observer to different estimation problems: reduced-order state observer design and parameter estimation in linear and nonlinear settings.

III. Main result

First, the conditions of existence of a static relationship between solutions of the systems (1) and (2) should be established. To this end, the invariant manifold method [11] will be adapted to the present problem to obtain a simple linear interconnection between the solutions of (1), (2).

A. Steady-state estimation

Proposition 1. *Assume that there exist $\Pi(t)\in\mathbb{R}^{q\times n}$ and $\Upsilon(t)\in\mathbb{R}^{q\times q}$ such that $\Upsilon(t)D_1(t)=0$, and also matrices $S_0(t)$, $S_1(t)$, $B(t)$, $O(t)$, $\forall t\in\mathbb{R}_+$, that satisfy the following linear equalities*[1]*:*

$$J\left(\Pi+\Upsilon D_0\right)=H,\tag{3}$$

and

$$\begin{aligned}&S_0\left(\Pi+\Upsilon D_0\right)+\left(B-\dot{\Upsilon}\right)D_0-\left(\Pi+\Upsilon D_0\right)A_0-\dot{\Pi}-\Upsilon\dot{D}_0=0,\\&BD_1-\left(\Pi+\Upsilon D_0\right)A_1+S_1=0,\\&\left(\Pi+\Upsilon D_0\right)Q=O.\end{aligned}\tag{4}$$

Then

$$\omega(t)=\Pi(t)x(t)+\Upsilon(t)y(t),\ \forall t\in\mathbb{R}_+\tag{5}$$

for any $x(0)\in\mathbb{R}^q$ and $\omega(0)=\Pi(0)x(0)+\Upsilon(0)y(0)$.

Proof. Let us check that $\omega(t)=\Pi(t)x(t)+\Upsilon(t)y(t)$, $\forall t\in\mathbb{R}_+$, is a solution of the system (1), (2). Taking the derivative of (5) and using equations (1), (2), since $\Upsilon D_1=0$, we have the following equality:

$$\begin{aligned}&S_0(\Pi+\Upsilon D_0)x+S_1f\left(J(\Pi x+\Upsilon y)\right)+B\left(D_0x+D_1f\left(Hx\right)\right)+Ou\\&=\left(\Pi+\Upsilon D_0\right)\left(A_0x+A_1f\left(Hx\right)+Qu\right)+\dot{\Pi}x(t)+\dot{\Upsilon}y\\&\quad+\Upsilon\left(\dot{D}_0x+\dot{D}_1f\left(Hx\right)\right).\end{aligned}$$

Also $\dot{\Upsilon}D_1=-\Upsilon\dot{D}_1$, so substituting (3) leads to the relation

$$\begin{aligned}&\left(S_0(\Pi+\Upsilon D_0)+(B-\dot{\Upsilon})D_0-(\Pi+\Upsilon D_0)A_0-\dot{\Pi}-\Upsilon\dot{D}_0\right)x\\&=\left(BD_1-(\Pi+\Upsilon D_0)A_1-S_1\right)f\left(Hx\right)+\left((\Pi+\Upsilon D_0)Q-O\right)u,\end{aligned}$$

which is satisfied thanks to (4). □

Since the observer (2) of the system (1) has the same shape of nonlinearity, under suitable interconnections among the matrices given in Proposition 1, the obtained relation between solutions is linear, reducing the complexity of analysis significantly and opening space for many applications.

B. Observer convergence to estimate

In Proposition 1, only the existence of relation $\omega(t)=\Pi(t)x(t)+\Upsilon(t)y(t)$ is proven for all $t\in\mathbb{R}_+$. Whether this relation is attracting for (1), (2) or not should be established in the convergence analysis.

Assumption 1. *There exist Π and Υ such that $\Upsilon D_1=0$ and the equalities (3) and (4) are satisfied for (1), (2).*

Consider the following dynamical system with a copy dynamics of (2):

$$\dot{z}(t)\ =\ S_0(t)z(t)+S_1(t)f(J(t)z(t))+B(t)y(t)+O(t)u(t),$$

Let us introduce the error $e:=\omega-z$ between two solutions of (2), initiated for different initial conditions, with the same inputs (y and u). Then we have the following dynamics:

$$\dot{e}(t)=S_0(t)e(t)+S_1(t)\left(f\left(J(t)\omega(t)\right)-f\left(J(t)z(t)\right)\right).\tag{6}$$

Now we can state a theorem about the convergence of the observer.

Theorem 1. *Let Assumption 1 be satisfied. Assume there exist $F(t)\in\mathbb{R}^{q\times k}$ and $W(t)=W(t)^\top\in\mathbb{R}^{q\times q}$ such that*

$$e^\top F\left(f\left(J\omega\right)-f\left(Jz\right)\right)\le e^\top We,$$

[1]Throughout the paper, we simplify the notation for matrices by writing, for instance, Π in place of $\Pi(t)$ when the time-dependency was once defined, unless the opposite is mentioned.

for all ω, $z \in \mathbb{R}^q$ and $e = \omega - z$, and that there exist $P(t) = P(t)^\top \in \mathbb{R}^{q\times q}$, $\Xi(t) = \Xi(t)^\top \in \mathbb{R}^{q\times q}$ such that the LMIs

$$\alpha_1 I_n \le P, \Xi \quad P \le \alpha_2 I_n, \quad PS_1 = F,$$
$$\dot{P} + S_0^\top P + PS_0 + 2W + \Xi < 0$$

have a solution for some $0 < \alpha_1 < \alpha_2 < +\infty$. Then, the system (6) is globally asymptotically stable and (2) is globally convergent.

Proof. Assumption 1 ensures the existence of an estimate, so let us select a Lyapunov function candidate $V(t, e(t)) = e(t)^\top P(t) e(t)$, where $P(t)$ is given in the conditions of the theorem. Then its derivative for (6) takes the form:

$$\dot{V} = e^\top (S_0^\top P + PS_0 + \dot{P})e + 2e^\top(t)PS_1\left(f(J\omega) - f(Jz)\right)$$
$$\le e^\top\left(S_0^\top P + PS_0 + \dot{P} + 2W + \Xi\right)e - e^\top \Xi e,$$

where $F = PS_1$ and W is given in the formulation of the theorem. According to the imposed conditions the top expression is non-positive and:

$$\|e(t)\| \le \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-0.5\frac{\lambda_{\min}(\Xi)}{\lambda_{\max}(P)}t}\|e(0)\|$$

for all $t \ge 0$. □

The conditions presented in Theorem 1 are given for an illustration, and any other conditions for convergence to zero and stability of e in (6) can be used.

IV. Conventional observer applications

Let us demonstrate how the generic results presented in the previous section can be used in several popular estimation scenarios.

A. Linear reduced-order observer

The first application of the result presented in the previous section, is a reduced-order observer for the linear case, where $S_1 = 0, D_1 = 0$, and all the other matrices are known and constant in (1) (the time-invariance has been imposed to simplify presentation and comparison). Then we have an ordinary LTI system of the form:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + Qu(t), \ t \in \mathbb{R}_+, \\ y(t) = D_0 x(t). \end{cases} \tag{7}$$

As in a classical problem of reduced-order observer design, we can present our state $x(t)$ with new variables $y(t) \in \mathbb{R}^p$ and $w(t) \in \mathbb{R}^{n-p}$ as follows:

$$\begin{pmatrix} y(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} D_0 \\ \Pi \end{pmatrix} x(t) \tag{8}$$

where $y(t)$ represents a set of directly measured state variables and $w(t)$ represents a set of unmeasured states, correspondingly. The task is then to build an observer for w, which will effectively reduce the dynamic order of (2) from n to $n-p$. Therefore, consider the following LTI system (the respective presentation of (2)):

$$\dot{\omega}(t) = S_0\omega(t) + By(t) + Ou(t), \tag{9}$$

where $\omega(t) \in \mathbb{R}^{n-p}$ is the observer state, and $S_0 \in \mathbb{R}^{(n-p)\times(n-p)}, B \in \mathbb{R}^{(n-p)\times p}, O \in \mathbb{R}^{(n-p)\times m}$ are constant matrices to be determined.

Then, Proposition 1 can be applied for the considered case. If the following matrix equalities are verified:

$$S_0(\Pi + \Upsilon D_0) + BD_0 - (\Pi + \Upsilon D_0)A_0 = 0, \tag{10}$$
$$(\Pi + \Upsilon D_0)Q = O,$$

then there exists a solution

$$\omega(t) = \Pi x(t) + \Upsilon y(t), \ \forall t \in \mathbb{R}_+ \tag{11}$$

for any $x(0) \in \mathbb{R}^q$ and $\omega(0) = \Pi x(0) + \Upsilon y(0)$, connecting (7) and (9). Applying Assumption 1 to (11), (7) and (9), we can use Theorem 1 (we ask for the existence of a positive definite symmetric matrix $P \in \mathbb{R}^{p\times p}$ such that $S_0^\top P + PS_0 < 0$) to show that (9) is globally asymptotically stable and it is, in fact, a global asymptotic observer for w, making it a reduced-order observer for (7). This conventional result is well-known, and it was first presented in [14]. The purpose of considering such a case is to demonstrate that the traditional result for linear systems can be obtained through the idea presented in Section III.

B. Linear regression

Another well-known and commonly used in practice estimation problem is the parameter linear regression. The application of our result is less intuitive in this setting than previously. However, it fits the problem statement rather well. In order to demonstrate it, let us consider a simple linear regression equation:

$$y(t) = D_0(t)x, \tag{12}$$

where $y(t)$ is a measured output, $D_0(t)$ is a regression matrix and x is the vector of unknown parameters to be estimated, which in our case is a state-vector. To apply Proposition 1, we consider $A_0 = 0$, $A_1 = 0$, $D_1 = 0$, $Q = 0, \forall t \in \mathbb{R}_+$ in (1) and we represent (12) in the required form (2) with $S_1 = 0$, $O = 0, \forall t \in \mathbb{R}_+$:

$$\dot{\omega}(t) = S_0(t)\omega(t) + B(t)y(t), \tag{13}$$

Then, we can choose $\Pi = I$ as constant identity matrix, $\Upsilon = 0, \forall t \in \mathbb{R}_+$, leading to:

$$\omega(t) = x(t), \tag{14}$$

which means that the system (13) is an observer for $x(t)$. Applying Proposition 1, we have equality:

$$S_0 = -BD_0. \tag{15}$$

Since we need to choose the matrices S_0 and B, let us assign $B := \Gamma D_0^\top$, where Γ is a nonsingular matrix, and can substantiate $S_0 = -\Gamma D_0^\top D_0$. Substituting in (13), it results in a conventional gradient estimator:

$$\dot{\omega}(t) = \Gamma D_0^\top(t)\left(y(t) - D_0(t)\omega(t)\right). \tag{16}$$

For asymptotic convergence of this observer we need a standard additional assumption:

Assumption 2. *The matrix $D_0(t)$ is PE, for $\forall t \in \mathbb{R}_+$ [5].*

Remark 1. There exist some techniques to relax the PE condition using different estimation algorithms, such as DREM (see for example [1]). However, to demonstrate our approach in this paper, we prefer to keep Assumption 2 for the sake of brevity.

Let Assumption 1 and Assumption 2 be satisfied, then applying Theorem 1 to (14), (12) and (13), we obtain the convergence of the estimator to the state x. This is a well-known result for parameter estimation in linear regression (see for instance [13]). Despite the result not being novel, this application demonstrates the wide range of applicability of the approach presented in Section III to estimation problems.

V. Nonlinear applications

Let us consider less investigated nonlinear scenario of applications, presented above.

A. Nonlinear reduced-order observer

Consider (1) and (2) with constant matrices for simplicity, and let $D_1 = 0$. We have nonlinear system:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + A_1 f(Hx(t)) + Qu(t), \ t \in \mathbb{R}_+, \\ y(t) = D_0 x(t), \end{cases} \tag{17}$$

where $x(t) \in \mathbb{R}^n$ is a full state, $y(t) \in \mathbb{R}^p$ is measured part of the state and A_0, A_1, Q, H are matrices of corresponding dimensions. Our goal is to design a reduced-order observer for (17) of a smaller dimension $q = n - p$:

$$\dot{\omega}(t) = S_0\omega(t) + S_1 f(J\omega(t)) + By(t) + Ou(t), \tag{18}$$

where $\omega \in \mathbb{R}^q$, and constant matrices are of corresponding dimension. All of them have to be defined using the presented method. Proposition 1 gives us the conditions of existence of a solution:

$$\omega(t) = \Pi x(t) + \Upsilon y(t) = (\Pi + \Upsilon D_0)x(t) = Zx(t), \tag{19}$$

where $Z \in \mathbb{R}^{q\times n}$ is a constant rectangular matrix, connecting solutions of initial system (17) and observer (18). Then, equations (3) and (4) in our case are as follows:

$$\begin{aligned} JZ &= H, \\ S_0 Z &= ZA_0 - BD_0, \\ S_1 &= ZA_1, \\ O &= ZQ. \end{aligned}$$

Having Z, the above equations can be solved with respect to S_0, S_1, B, O and J. Theorem 1 can be directly applied with a constant matrix P in order to prove the convergence of the observer (18) to a hyperplane given by (19). Since the order of ω is lower than the one of x, we can say that (18) is an asymptotical reduced-order observer.

Remark 2. The same result can be derived for time-varying matrices, the difference will lay only in matrix equations from Proposition 1 and in the form of $S_0(t)$ and $S_1(t)$.

The results on reduced-order observer design for nonlinear systems given in [17] are local and based on linearization analysis, while the general results presented in [11] cover the considered case, but they are less constructive since require a solution of PDE.

B. Nonlinear regression

Similarly to linear equation (12), we can consider the following form of the system (1), where $A_0 = 0, A_1 = 0, Q = 0$:

$$y(t) = D_0(t)x + D_1(t) f(H(t)x),$$

where $y(t)$ is measured output vector, $D_0(t)$ is a regression matrix in linear part and $D_1(t), H(t)$ are regression matrices in nonlinear part, x is a vector of unknown parameters. As in the linear case we have corresponding observer:

$$\dot{\omega}(t) = S_0(t)\omega(t) + S_1(t) f(J(t)\omega(t)) + B(t)y(t), \tag{20}$$

and we put $\Pi = I, \Upsilon = 0$, so we are looking for the solution in the form:

$$\omega(t) = x(t).$$

Using Proposition 1 we obtain following equalities:

$$\begin{aligned} J &= H, \\ S_0 + BD_0 &= 0, \\ S_1 + BD_1 &= 0. \end{aligned}$$

Finally, substituting it to (20) with a nonsingular matrix Γ of the appropriate dimension, we have:

$$\dot{\omega}(t) = -B(t)D_0(t)\omega(t) - B(t)D_1(t) f(J(t)\omega(t)) + B(t)y(t).$$

Let $B(t) = \Gamma D_0(t)^\top$, we have:

$$\dot{\omega}(t) = -\Gamma D_0^\top D_0\omega(t) - \Gamma D_0^\top D_1 f(J\omega(t)) + \Gamma D_0^\top y(t), \tag{21}$$

Let Assumption 1 and 2 be satisfied, then according to Theorem 1, (21) is an asymptotic observer for $x(t)$, (provided that the imposed there restrictions on nonlinearity are verified).

To the best of our knowledge, nonlinear regression for the considered class of systems is a novel result.

Nevertheless, the current section demonstrates an advantage of the generality of the approach considered in Section III, which allows investigating a rather wide range of both linear and nonlinear problems.

VI. Examples

Example 1. Let us consider a two mass-spring system with nonlinear stiffness, the dynamics of which can be expressed as follows:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -k_1(x_1 - x_3) - k_2(x_1 - x_3)^3 - a_1(x_2 - x_4), \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = k_1(x_1 - x_3) + k_2(x_1 - x_3)^3 + a_1(x_2 - x_4) - k_3 x_3 - a_2 x_4 + \sin(t), \\ y_1 = x_1, \\ y_2 = x_2 \end{cases} \tag{22}$$

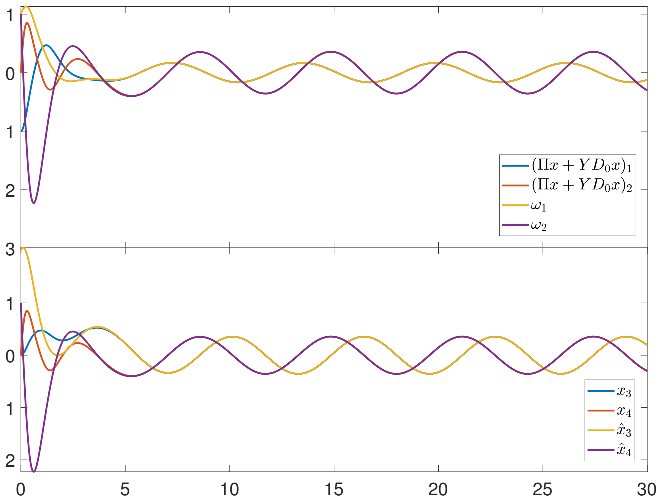

Fig. 1. Example 1.

Presenting the system in the form (1), we have:

$$A_0=\begin{pmatrix}0 & 1 & 0 & 0\\ -k_1 & -a_1 & k_1 & a_1\\ 0 & 0 & 0 & 1\\ k_1 & a_1 & -k_1-k_3 & a_1-a_2\end{pmatrix},\quad A_1=\begin{pmatrix}0\\ -k_2\\ 0\\ k_2\end{pmatrix},$$

$$f(Hx(t))=(Hx(t))^3,\quad Q=\begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}^\top,$$

$$D_0=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix},\quad H=\begin{pmatrix}1 & 0 & -1 & 0\end{pmatrix},\quad D_1=0,$$

where the states x_1 and x_2 correspond to the position and the velocity of the first mass, assumed to be measured. The states x_3 and x_4 represent the position and the velocity of the second mass, which we need to estimate. The signal $u(t)=\sin(t)$ is the periodic force, applied to the second mass to excite the system. Thus, the task is to build an observer of the form (18), (19) for the states x_3 and x_4 using Proposition 1. We have the following LMEs:

$$\begin{aligned} J(\Pi+\Upsilon D_0)&=H, \\ S_0(\Pi+\Upsilon D_0)&=(\Pi+\Upsilon D_0)A_0-BD_0,\\ S_1&=(\Pi+\Upsilon D_0)A_1,\\ O&=(\Pi+\Upsilon D_0)Q.\end{aligned} \tag{23}$$

We can assign Π and Υ having full row rank (the matrix Π should have left inverse with respect to the unmeasured state components of (22), while Υ just describes the utilization of the output variables) as, for example:

$$\Pi=\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1,\end{pmatrix}\quad \Upsilon=\begin{pmatrix}\upsilon_1 & 0\\ 0 & \upsilon_2\end{pmatrix}.$$

Therefore, we have 16 equations in total, with 16 unknowns. Solving the equations for chosen in simulation values of the coefficients $(k_1=3, k_2=3, k_3=0.6, a_1=0.6, a_2=2)$, we obtain:

$$S_0=\begin{pmatrix}0 & 1\\ -3.6 & -2.6\end{pmatrix},\quad S_1=\begin{pmatrix}0\\ 3\end{pmatrix},$$

$$J=\begin{pmatrix}-1 & 0\end{pmatrix},\quad O=\begin{pmatrix}0\\ 1\end{pmatrix},\quad B=\begin{pmatrix}0 & -1\\ -0.6 & 0.6\end{pmatrix},\quad \Upsilon=\begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}.$$

Then, we have the dynamics of ω, which must satisfy the LMIs in Theorem 1. From the given dynamics, we have a condition:

$$e^\top F\left(z_1^3-\omega_1^3\right)\le(F_1e_1+F_2e_2)\left(-e_1(z_1^2+\omega_1 z_1+\omega_1^2)\right)\le e^\top We,$$

which has to be satisfied either globally, or at least locally in $z,\omega\in\mathbb{R}^2$ Consider, for example the following matrices:

$$F=\begin{pmatrix}0\\ 1\end{pmatrix},\quad W=\begin{pmatrix}0 & -0.5\\ -0.5 & 0\end{pmatrix},$$

and the solution of LMIs given by the solver is:

$$P=\begin{pmatrix}2.2 & 0\\ 0 & 0.33\end{pmatrix},\quad \Xi=\begin{pmatrix}0 & 0\\ 0 & 0.8667\end{pmatrix}.$$

Finally, we build an observer for states x_3 and x_4 from the expression $\omega=\Pi x+\Upsilon D_0 x$:

$$\begin{aligned}\hat{x}_3&=\frac{\omega_1-\upsilon_1 x_1}{p_1}=\omega_1+x_1,\\ \hat{x}_4&=\frac{\omega_2-\upsilon_4 x_2}{p_4}=\omega_2.\end{aligned}$$

Figure 1 demonstrates the convergence of the observer for unmeasured states, which successfully reduces the dimension of the observation problem from 4 to 2, in the considered case.

VII. Conclusion

The results presented in this paper simplify designing (reduced-order) observers for a particular class of systems, avoiding solutions of PDEs which arise in the conventional invariant manifold methodology (see [2]). The resulting solution is explicit and more constructive than existing results on nonlinear observer design. We have shown two possible applications of our approach: nonlinear reduced-order observer and nonlinear regression. An example representing a mechanical system demonstrates the applicability of our result in this work.

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