On steady-state based reduced-order observer design for interlaced nonlinear systems

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***Abstract*— This paper proposes an analytical expression for a nonlinear mapping between steady-state solutions of certain types of nonlinear interconnected systems. This mapping is found using tools from the theory of output regulation for systems presented in lower-triangular or upper-triangular canonical forms. Next, this mapping helps design an excitation input and a corresponding reduced-order observer for interlaced systems, a combination of both upper- and lower-triangular subsystems. A proposed global observer is proved to be robust to additive disturbance and measurement noise by applying the method of Lyapunov function. An example involving mass-spring system demonstrates the efficiency of our approach.**

I. Introduction

The moment theory for nonlinear systems [16], [2], based on the theory of output regulation [5], [8] and center manifold theory, asserts that under certain assumptions, any nonlinear system must have a *central* manifold, an invariant subspace containing the steady-state solution of the system, that can be used to replicate its behavior (*i.e.*, by frequency response). However, despite the solid theoretical background, the process of determining of steady-state solutions of nonlinear systems is still dependent on a system's form.

This paper is addressing the problem of finding the steady-state response for particular types of nonlinear systems inspired by recursive procedures, such as backstepping and forwarding [19]. We propose a way of determining steady-state solutions for systems based on the diagonal structure of its model.

The main idea of backstepping, introduced in [11], consists of using the lower-triangular form of a system to make a step-by-step output passivation through time-derivative, starting from the first equation to the last one, where the stabilizing input is designed (*e.g.* see [13], [12], [6], [10]). Forwarding is a more recent method of stabilization [18], which utilizes the upper-triangular form of systems. It has a similar iterative idea to backstepping, which works in the opposite direction: it uses the integration starting from the input and goes forward (*e.g.* see [23], [3]). The *interlaced systems* introduced in [19] combine both types of triangular dynamics in their subsystems. It is a large class of models, which have restrictions only on the structure of their feedback and feedforward interconnections.

This paper uses techniques similar to backstepping or forwarding and derives the steady-state solutions of such an interlaced system. One of our goals is to design a robust reduced-order observer with the corresponding input to the system. This result could significantly improve the observer design for the considered classes of nonlinear systems without the hypothesis about uniform observability.

The outline of the paper is as follows. Notation and definitions used throughout the paper are introduced in Section II, along with the problem statement in Section III. The results, including the methods for analytical calculation of the steady-state solutions for two types of canonical forms of nonlinear systems, are presented in Section IV. The observer design is given in Section V. In Section VI, the efficiency of the proposed solutions is illustrated in the example.

II. Preliminaries

A. Notation

- The set of real numbers is denoted by $\mathbb{R}$ and we write $\mathbb{R}_+ := \{t \in \mathbb{R} : t \geq 0\}$.
- The spaces of real matrices of dimension $n \times m$ and real vectors of dimension n are denoted by $\mathbb{R}^{n\times m}$ and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$, respectively.
- For a Lebesgue measurable function $u : \mathbb{R}_+ \to \mathbb{R}^n$, define the norm $\|u\|_{[t_1,t_2)} = \operatorname{ess\,sup}_{t\in[t_1,t_2)} \|u(t)\|$ for $[t_1,t_2) \subset \mathbb{R}_+$, where $\|\cdot\|$ refers to the Euclidian norm in $\mathbb{R}^n$. We denote by $\mathcal{L}_\infty^n$ the set of functions $u : \mathbb{R}_+ \to \mathbb{R}^n$ such that $\|u\|_\infty := \|u\|_{[0,+\infty)} < +\infty$.
- For a vector $\xi = (\xi_1 \ \dots \ \xi_n)^\top \in \mathbb{R}^n$ and $i \in \{1,\dots,n\}$, the subvector $(\xi_1 \ \xi_2 \ \dots \ \xi_i)^\top \in \mathbb{R}^i$ is denoted by $\underline{\xi}_i$, and correspondingly, the subvector $(\xi_i \ \xi_{i+1} \ \dots \ \xi_n)^\top \in \mathbb{R}^{n-i+1}$ is denoted by $\bar{\xi}_i$.

B. Theory of moments

Let us consider a nonlinear continuous-time system:

$$\begin{aligned} \dot{x}(t) &= f(x(t),u(t)),\ \forall t \in \mathbb{R}_+, \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $x : \mathbb{R}_+ \to \mathbb{R}^n$ is the state function, $u : \mathbb{R}_+ \to \mathbb{R}^m$ is the input signal, and $y : \mathbb{R}_+ \to \mathbb{R}^p$ is the output; $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are smooth functions with $f(0,0) = 0$ and $h(0) = 0$. Consider a signal generator system given by:

$$\begin{aligned} \dot{\omega}(t) &= s(\omega(t)),\ \forall t \in \mathbb{R}_+, \\ u(t) &= \ell(\omega(t)), \end{aligned} \tag{2}$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}^q$ is the state; $s : \mathbb{R}^q \to \mathbb{R}^q$, $\ell : \mathbb{R}^q \to \mathbb{R}^m$ are smooth functions with $s(0) = 0$ and $\ell(0) = 0$. The

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interconnected system is given by (1) and (2)[1]:

$$\dot{x} = f(x, \ell(\omega)), \quad \dot{\omega} = s(\omega), \quad y = h(x). \tag{3}$$

We assume the following conditions:

Assumption 1. *[17] The system (1) is minimal and the zero equilibrium is locally exponentially stable. The signal generator system (2) is observable and neutrally stable.*

The minimal condition of the considered system implies its *observability* and *accessibility*.

Lemma 1. *[17] Consider the system (1) and the signal generator system (2). Suppose Assumption 1 holds. Then there is a mapping $\pi : \mathbb{R}^q \to \mathbb{R}^n$, locally defined in a neighborhood $\mathcal{W} \subset \mathbb{R}^q$ of the origin, with $\pi(0) = 0$, which solves the partial differential equation:*

$$\frac{\partial \pi}{\partial w}(w)s(w) = f(\pi(w), \ell(w)), \tag{4}$$

for all $w \in \mathcal{W}$. In addition, the steady-state response of the system (3) is $x^{\mathrm{ss}}(t) = \pi(\omega(t))$ for any $x(0)$ and $\omega(0)$ sufficiently small.

Note that other results and definitions accompanying the moment theory, along with the definitions of observability and accessibility for nonlinear systems, can be found in [17] (see Def. 2.1-2.9).

C. Convergent systems

The (4) is a first order quasi-linear partial differential equation. A solution of this equation represents an invariant trajectory for (3), but if, in addition, we ask for such a solution to be attractive for the surrounding trajectories in some region, then we come to the concept of convergent dynamics [15].

Definition 1. [1] The system (1) is said to be *incrementally input-to-state stable* (δISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for any $t \geq 0$, any $\kappa_1, \kappa_2 \in \mathbb{R}^n$ and any couple of signals $d_1, d_2 \in \mathcal{L}_\infty^m$, the following holds:

$$\|x(t,\kappa_1,d_1) - x(t,\kappa_2,d_2)\| \leq \beta(\|\kappa_1 - \kappa_2\|, t) + \gamma(\|d_1 - d_2\|_\infty).$$

If this property is satisfied, all trajectories of (3) converge to a solution of (4) and stay close to it in the case of additive perturbations in the input channel of (1).

D. Existence of solutions for first order quasi-linear PDEs

Consider a quasi-linear partial differential equation of the first order [4] (a variant of (4)):

$$\frac{\partial \chi}{\partial w}\sigma(\chi, w) = \phi(\chi, w), \tag{5}$$

where $\chi : \mathbb{R}^q \to \mathbb{R}$ is a function, $\sigma : \mathbb{R}^{q+1} \to \mathbb{R}^q$ and $\phi : \mathbb{R}^{q+1} \to \mathbb{R}$ are real analytic functions ($\sigma(0,0) = 0$ and $\phi(0,0) = 0$), with initial or boundary conditions

$$\chi(w) = 0, \ \forall w \in \Gamma = \{w \in \mathcal{W} : w_q = 0\}, \tag{6}$$

[1]Throughout the paper, we simplify the notation by writing, for instance, x in place of $x(t)$ when the time-dependency is clear.

where $\mathcal{W} \subset \mathbb{R}^q$ is a neighborhood of the origin and w_q denotes the q-th component of $w \in \mathbb{R}^q$. The Cauchy problem (5), (6) is called *non-characteristic* if

$$\sigma_q(0, w) \neq 0, \ \forall w \in \Gamma \setminus \{0\},$$

where the origin is excluded since in such a case the trivial solution $\chi(0) = 0$ exists by construction. The conditions for the existence of a solution χ to (5), (6) are given by the Cauchy-Kowalevski Theorem:

Theorem 1. *[22] The Cauchy problem (5), (6) has a unique real analytic solution χ on $\mathcal{W}$ if the Cauchy problem is non-characteristic.*

In the case when σ and ϕ are continuously differentiable functions, analogues of this theorem can be found, for example, in [4], [7] providing a continuously differentiable solution χ.

III. Problem Statement

Consider an interconnection of two nonlinear systems one of which is in a form similar to (1) but with disturbance and measurement noise, and the second one being (2):

$$\begin{cases} \dot{x} = F(x) + G(x, u) + d, \\ y = h(x) + v; \end{cases} \tag{7}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ are smooth or real analytic maps, and the remaining functions are as in (1) with $m = 1$; $F(0) = 0$, $G(\cdot, 0) = 0$; $d \in \mathcal{L}_\infty^n$ is an unknown essentially bounded disturbance, $v \in \mathcal{L}_\infty^p$ is a measurement noise signal.

This paper aims at the design of a robust reduced-order observer for classes of lower- or upper-triangular functions F and G in (7) using tools from the moment theory for nonlinear systems. The derived expressions of steady-state solutions will be used next to design a robust reduced-order observer to reconstruct the state of (7), (2). We also assume that the form of the input signal ℓ can be chosen to ensure observability or detectability of (7), (2).

IV. The Steady-State Response of Feedback and Feedforward Systems in a Disturbance-Free Case

Two particular classes of F and G are studied in this section.

A. Feedback (lower-triangular form) case

A particular case of the system (7) is given by a nonlinear single-input system in the canonical feedback form ($d = 0$):

$$\begin{aligned} \dot{z} &= \eta(z, \xi_1), && (8) \\ \dot{\xi}_1 &= f_1(\xi_1, z) + g_1(\xi_1, z)\xi_2, && (9) \\ &\vdots \\ \dot{\xi}_{\nu-1} &= f_{\nu-1}(\underline{\xi}_{\nu-1}, z) + g_{\nu-1}(\underline{\xi}_{\nu-1}, z)\xi_\nu && (10) \\ \dot{\xi}_\nu &= f_\nu(\xi, z) + g_\nu(\xi, z)u, && (11) \end{aligned}$$

with the same signal generator system as in (2), where $z : \mathbb{R}_+ \to \mathbb{R}^r$ is the zero dynamics state, $\xi = (\xi_1 \ \ldots \ \xi_\nu)^\top \in \mathbb{R}^\nu$

and $x = (z^\top\ \xi^\top)^\top \in \mathbb{R}^{\nu+r}$, $n = \nu + r$; and the functions $\eta : \mathbb{R}^{r+1} \to \mathbb{R}^r$, f_i, $g_i : \mathbb{R}^{r+i} \to \mathbb{R}$ for $i = 1, \dots, \nu$ are smooth. The conditions providing a representation of (7) in the form (8)-(11) can be checked in [19].

Remark that the input u of (8)-(11) appears solely in the state equation for ξ_ν. Hence, inspired by the backstepping idea, if $\xi_1 = \pi_1(\omega)$ is fixed, then an input function $\ell(\omega)$ can be designed in such a way that it assigns π_1 as the steady-state solution of this interconnected system. Thus, using the theory of output regulation, we can formulate the conditions for the existence of a steady-state solution for (2)-(11):

Proposition 1. *For any smooth function $\pi_1 : \mathbb{R}^q \to \mathbb{R}$, let a smooth function $\rho : \mathbb{R}^q \to \mathbb{R}^r$ be the steady-state solution of (8) under substitution $\xi_1 = \pi_1(\omega)$. Then there always exists the input function $\ell : \mathbb{R}^q \to \mathbb{R}$ such that $\pi(\omega) = (\pi_1(\omega)\ \dots\ \pi_\nu(\omega))^\top$ is the steady-state solution of the interconnected system (2)-(11) provided that $g_i(\underline{\pi}_i(\omega), \rho(\omega)) \neq 0$ for all $i = 1, \dots, \nu$ and $\omega \in \mathbb{R}^q$.*

Proof. We can set $\xi_1 = \pi_1(\omega)$, which is the first steady-state solution of the system (11), then $z = \rho(\omega)$ is the corresponding steady-state of the zero dynamics (8). To find the remaining components of π and the respective input function ℓ, let us use the output regulation theory, which states that the equation (4) must be satisfied. Consequently, we obtain

$$
\begin{aligned}
\frac{\partial \pi_1(\omega)}{\partial \omega} s(\omega) &= f_1(\pi_1(\omega), \rho(\omega)) + g_1(\pi_1(\omega), \rho(\omega)) \pi_2(\omega), \\
&\vdots \\
\frac{\partial \pi_{\nu-1}(\omega)}{\partial \omega} s(\omega) &= f_{\nu-1}(\underline{\pi_{\nu-1}}(\omega), \rho(\omega)) \\
&\quad + g_{\nu-1}(\underline{\pi_{\nu-1}}(\omega), \rho(\omega)) \pi_\nu(\omega), \\
\frac{\partial \pi_\nu(\omega)}{\partial \omega} s(\omega) &= f_\nu(\pi(\omega), \rho(\omega)) + g_\nu(\pi(\omega), \rho(\omega)) \ell(\omega).
\end{aligned}
$$

Since the considered form of the system (11) is lower-triangular (feedback based), and only the last equation depends on the input, the components π_i and the input ℓ can be found step by step from the top system as follows:

$$
\begin{aligned}
\pi_2(\omega) &= \frac{\frac{\partial \pi_1(\omega)}{\partial \omega} s(\omega) - f_1(\pi_1(\omega), \rho(\omega))}{g_1(\pi_1(\omega), \rho(\omega))}, \\
&\vdots \\
\pi_\nu(\omega) &= \frac{\frac{\partial \pi_{\nu-1}(\omega)}{\partial \omega} s(\omega) - f_{\nu-1}(\underline{\pi_{\nu-1}}(\omega), \rho(\omega))}{g_{\nu-1}(\underline{\pi_{\nu-1}}(\omega), \rho(\omega))}, \\
u = \ell(\omega) &= \frac{\frac{\partial \pi_\nu(\omega)}{\partial \omega} s(\omega) - f_\nu(\pi(\omega), \rho(\omega))}{g_\nu(\pi(\omega), \rho(\omega))},
\end{aligned}
$$

where $g_i(\underline{\pi}_i(\omega), \rho(\omega)) \not\equiv 0$ according to the imposed conditions. □

Note that the conditions of Proposition 1 do not imply any attractivity of the derived steady-state solution $x(t) = \pi(\omega(t))$ for (2)-(11), only its existence is established.

Remark 1. In Proposition 1, it is assumed that the zero dynamics has a well-defined respective steady-state solution ρ. Such an additional hypothesis has a counterpart in the backstepping method, where for the system stabilization, it is usually assumed that the zero dynamics is input-to-state stable [19]. Here, to find steady-state solutions for ξ, we need to settle the same property for z.

B. Feedforward (upper-triangular form) case

Let us consider a nonlinear single-input system in the feedforward canonical form ($d = 0$) [19]:

$$
\begin{aligned}
\dot{\zeta}_1 &= \widetilde{f}_1(\zeta_1) + \widetilde{g}_1(\overline{\zeta}_2, z, u), &(12)\\
&\vdots \\
\dot{\zeta}_{\mu-1} &= \widetilde{f}_{\mu-1}(\zeta_{\mu-1}) + \widetilde{g}_{\mu-1}(\zeta_\mu, z, u), &(13)\\
\dot{\zeta}_\mu &= \widetilde{f}_\mu(\zeta_\mu) + \widetilde{g}_\mu(u) &(14)\\
\dot{z} &= \widetilde{\eta}(z, \zeta_\mu, u) &(15)
\end{aligned}
$$

with the same signal generator (2), where $z : \mathbb{R}_+ \to \mathbb{R}^r$ is the zero dynamics state, $\zeta = \left(\zeta_1\ \dots\ \zeta_\mu\right)^\top \in \mathbb{R}^\mu$ and $x = (z^\top\ \zeta^\top)^\top$, $n = \mu + r$; the functions $\widetilde{\eta} : \mathbb{R}^{r+2} \to \mathbb{R}^r$, $\widetilde{f}_i : \mathbb{R} \to \mathbb{R}$ and $\widetilde{g}_i : \mathbb{R}^{\mu-i+r+1} \to \mathbb{R}$ are real analytic for $i = 1, \dots, \mu$ taking zero value for zero argument. Due to the upper-triangular structure of the system (12)-(15), the control propagates from the last equation for ζ_μ till ζ_1, allowing the steady-state solutions to be iteratively calculated. For instance, the equation (4) for the variable ζ_μ can be written as follows:

$$
\frac{\partial \pi_\mu(\omega)}{\partial \omega} s(\omega) = \widetilde{f}_\mu(\pi_\mu(\omega)) + \widetilde{g}_\mu(\ell(\omega)) \tag{16}
$$

provided that the control signal $u = \ell(\omega)$ is fixed (for a real analytic function ℓ). For a given s and any initial condition $\omega(0) \in \Gamma = \{w \in \mathcal{W} : w_q = 0\}$, where $\mathcal{W} \subset \mathbb{R}^q$ is a neighborhood of the origin, the signal generator (2) has a solution ω that is defined at least locally in time. Assuming that

$$
\forall \omega \in \Gamma \setminus \{0\} : s_q(\omega) \neq 0, \tag{17}
$$

there is a real analytic solution $\pi_\mu : \mathcal{W} \to \mathbb{R}$ of the output regulation equation (16) according to Theorem 1 (since (17) implies that (16) can be completed by initial conditions making this Cauchy problem non-characteristic). The same procedure can be repeated/forwarded for the rest of the system:

Proposition 2. *Let (17) be satisfied. For any real analytic $\ell : \mathbb{R}^q \to \mathbb{R}$, let π_μ be a solution of (16) and $\rho : \mathbb{R}^q \to \mathbb{R}^r$ be the steady-state solution of (15) under substitution $\zeta_\mu = \pi_\mu(\omega)$ and $u = \ell(\omega)$. Then there exists a real analytic steady-state mapping $\pi(\omega) = \left(\pi_1(\omega)\ \dots\ \pi_\mu(\omega)\right)^\top$, $\omega \in \mathcal{W}$, of the interconnected system (2),(12)-(15) with $\pi(0) = 0$.*

Proof. By choosing $\ell(\omega)$ such that $\pi_\mu(\omega)$ satisfies (16), we obtain $\rho(\omega)$ from (15). For the remaining $\pi_i(\omega)$, $i = 1, \dots, \mu - 1$, we have the following equations from (4):

$$
\begin{aligned}
\frac{\partial \pi_1(\omega)}{\partial \omega} s(\omega) &= \widetilde{f}_1(\pi_1(\omega)) + \widetilde{g}_1(\overline{\pi}_2(\omega), \rho(\omega), \ell(\omega)), \\
&\vdots \\
\frac{\partial \pi_{\mu-2}(\omega)}{\partial \omega} s(\omega) &= \widetilde{f}_{\mu-2}(\pi_{\mu-2}(\omega)) \\
&\quad + \widetilde{g}_{\mu-2}(\overline{\pi}_{\mu-1}(\omega), \rho(\omega), \ell(\omega)), \\
\frac{\partial \pi_{\mu-1}(\omega)}{\partial \omega} s(\omega) &= \widetilde{f}_{\mu-1}(\pi_{\mu-1}(\omega)) + \widetilde{g}_{\mu-1}(\pi_\mu(\omega), \rho(\omega), \ell(\omega)),
\end{aligned}
$$

which all yield the form of (16) and the same condition of being non-characteristic for the initial values $\pi_i(\omega) = 0$ for $\omega \in \Gamma$ (since the system is in the upper-triangular form, $\pi_1(\omega), \ldots, \pi_{\mu-1}(\omega)$ can be found iteratively from the last equation to the first). Then, according to Theorem 1, these first order quasi-linear PDEs have unique analytic solutions $\pi_i : \mathcal{W} \to \mathbb{R}$, $i = 1, \ldots, \mu - 1$. □

Here again, we need to assume the existence of the steady-state solution ρ for the zero dynamics (15), while in the forwarding method, the input-to-state stability hypothesis is used (compare to Remark 1 for the lower-triangular case). Similarly, Proposition 2 establishes the existence of the steady-state solutions, but not their attractiveness to surrounding solutions.

V. Observer design for interlaced systems

To show the utility of the presented results to the analytical derivation of a steady-state solution, consider the problem of state estimation for a nonlinear system. The considered model is a combination of both studied systems, the upper- and the lower-triangular ones (with disturbances d_i, v):

$$\begin{aligned}
\dot{\zeta}_1 &= \widetilde{f}_1(\zeta_1) + \widetilde{g}_1(\overline{\zeta}_2, z) + d_1, \\
&\vdots \\
\dot{\zeta}_{\mu-1} &= \widetilde{f}_{\mu-1}(\zeta_{\mu-1}) + \widetilde{g}_{\mu-1}(\overline{\zeta}_\mu, z) + d_{\mu-1}, \\
\dot{\zeta}_\mu &= \widetilde{f}_\mu(\zeta_\mu) + \widetilde{g}_\mu(\xi_1, z) + d_\mu \\
,\dot{\xi}_1 &= f_1(\underline{x}_{\mu+1}, z) + g_1(\underline{x}_{\mu+1}, z)\xi_2 + d_{\mu+1}, \\
\dot{\xi}_2 &= f_2(\underline{x}_{\mu+2}, z) + g_2(\underline{x}_{\mu+2}, z)\xi_3 + d_{\mu+2}, \\
&\vdots \\
\dot{\xi}_{\nu-1} &= f_{\nu-1}(\underline{x}_{\mu+\nu-1}, z) + g_{\nu-1}(\underline{x}_{\mu+\nu-1}, z)\xi_\nu + d_{\mu+\nu-1}, \\
\dot{\xi}_\nu &= f_\nu(x, z) + g_\nu(x, z)u + d_{\mu+\nu}, \\
\dot{z} &= \eta(z, \xi_1), \\
y &= h(z, x) + v.
\end{aligned} \tag{18}$$

A system in such a form is called interlaced. Variables $\zeta \in \mathbb{R}^\mu$ correspond to the state of the upper-triangular part, variables $\xi \in \mathbb{R}^\nu$ describe the state of the lower-triangular part; $x = (\zeta^\top \ \xi^\top)^\top \in \mathbb{R}^n$ and $n = \mu + \nu$, $z \in \mathbb{R}^r$ is the state of zero dynamics; $y \in \mathbb{R}^p$ is the output available for measurements. We assume that all nonlinear functions in (19) satisfy the requirements on regularity needed to use propositions 1 and 2, *i.e.*, they all are real analytic and admit the zero solution, h is globally Lipschitz continuous. The signal generator is considered in the linear form to simplify the observer design:

$$\dot{\omega} = A\omega, \ u = \ell(\omega), \tag{19}$$

where as before $\omega : \mathbb{R}_+ \to \mathbb{R}^q$, a neutrally stable matrix $A \in \mathbb{R}^{q\times q}$ is chosen in a way that the condition (17) is satisfied:

$$\forall \omega \in \Gamma \setminus \{0\} : [0 \ldots 0\ 1]A\omega \neq 0. \tag{20}$$

Our goal is to design a robust reduced-order observer for (19), (19) using steady-state solutions $\pi(\omega)$ and $\rho(\omega)$, as established before, and assuming that they describe the steady-state solutions of this system. Note that there is no restriction imposed on the uniform observability of (19) (*i.e.*, observability for all admissible inputs u). We assume below that the input function ℓ is properly set to ensure state estimation. In such a setup, it is enough to design an observer for the generator (19) and calculate the asymptotic estimates of the steady-state solutions.

Thus, we start with the next hypotheses:

Assumption 2. *Assume*

1) *the condition* (20) *holds;*
2) *for any real analytic $\pi_{\mu+1} : \mathbb{R}^q \to \mathbb{R}$ there is the corresponding steady-state solution $\rho : \mathbb{R}^q \to \mathbb{R}^r$ of the zero dynamics under substitution $\xi_1 = \pi_{\mu+1}(\omega)$.*

Under Assumption 2, the conditions of propositions 1 and 2 are satisfied for the system (19), (19), with $d = 0$. Due to the structure of (19), for any $\pi_{\mu+1}$ and the related ρ, the steady-state solution $\underline{\pi}_\mu$ can be derived following the forwarding guidelines in the proof of Proposition 2. Next, the steady-state solution $\overline{\pi}_{\mu+2}$ can be calculated following the backstepping procedure of Proposition 1, together with the input function $\ell : \mathbb{R}^q \to \mathbb{R}$. Thus, all steady-state solutions $\pi : \mathbb{R}^q \to \mathbb{R}^n$ can be derived, for which we assume the following:

Assumption 3. *Let* 1) *the function $\pi_{\mu+1}$ be chosen such that $h(\rho(\omega), \pi(\omega)) = C\omega$, where the pair (A, C) is detectable;*
2) *the system* (19) *be δISS for $u = \ell(\omega)$, for any solution ω of* (19) *with respect to the input d.*

Under these restrictions, the following theorem formulates the main result of the paper:

Theorem 2. *Let assumptions 2, 3 be satisfied, $u = \ell(\omega)$, and suppose there exist $P = P^\top \in \mathbb{R}^{q\times q}$, $Q = Q^\top \in \mathbb{R}^{q\times q}$, $\Gamma = \Gamma^\top \in \mathbb{R}^{p\times p}$ and $U \in \mathbb{R}^{q\times p}$ such that the linear matrix inequalities*

$$P > 0, \quad Q > 0, \quad \Gamma > 0,$$
$$\begin{pmatrix} A^\top P - C^\top U^\top + PA - UC + Q & -U \\ -U^\top & -\Gamma \end{pmatrix} \leq 0$$

have a solution. Then

$$\dot{\hat{\omega}} = A\hat{\omega} + L(y - C\hat{\omega}), \quad \hat{x} = \pi(\hat{\omega}), \quad \hat{z} = \rho(\hat{\omega}) \tag{21}$$

is a global robust asymptotic reduced-order observer for the system (19), (19) *with $L = P^{-1}U$, where $\hat{\omega}(t) \in \mathbb{R}^q$, $\hat{x}(t) \in \mathbb{R}^n$ and $\hat{z}(t) \in \mathbb{R}^r$ are the estimates of $\omega(t)$, $x(t)$ and $z(t)$, respectively.*

Proof. According to Assumption 2, the system (19) has steady-state solutions π and ρ for any $\pi_{\mu+1}$ with the corresponding input ℓ (by propositions 1 and 2). Assumption 3 states that the function $\pi_{\mu+1}$ was selected in a way that the measured output y is a linear function of the generator state ω while being projected on the steady-state solution, and in addition, $\pi(\omega)$ and $\rho(\omega)$ are steady-state solutions to which $x(t)$ and $z(t)$ converge when $t \to +\infty$, respectively, for the assigned input $u = \ell(\omega)$. Therefore, we can write that we measure $y = C\omega + \varepsilon + v$, where $\varepsilon : \mathbb{R}_+ \to \mathbb{R}^p$ is a bounded error representing the convergence of solutions of (19) to the

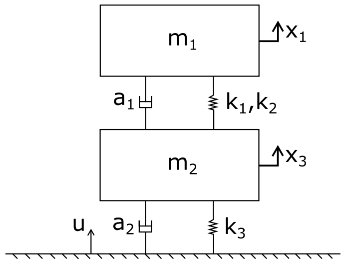

Fig. 1. Schematics for Example 1

steady-state ones, $\varepsilon = h(z,x) - h(\rho(\omega),\pi(\omega))$. To analyze the behavior of $e = \omega - \hat{\omega}$, whose dynamics take the form:

$$\dot{e} = (A - LC)e + L(\varepsilon + v),$$

let us select a quadratic Lyapunov function candidate $V(e) = e^\top Pe$, where P is given in a formulation of the theorem. Denote $\widetilde{v} = \varepsilon + v$, then we have:

$$\dot{V} = \begin{pmatrix} e \\ \widetilde{v} \end{pmatrix}^\top \begin{pmatrix} A^\top P - C^\top U^\top + PA - UC + Q & -U \\ -U^\top & -\Gamma \end{pmatrix} \begin{pmatrix} e \\ \widetilde{v} \end{pmatrix} - e^\top Qe + \widetilde{v}^\top \Gamma \widetilde{v}.$$

According to the formulation of the theorem, the matrix above is non-positive. Then, we obtain:

$$\dot{V} \le -e^\top Qe + \widetilde{v}^\top \Gamma \widetilde{v}.$$

After straightforward calculations, we can obtain the estimate on the behavior of e as follows:

$$\|e(t)\| \le \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} e^{-0.5\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}t} \|e(0)\| + \sqrt{\frac{\lambda_{\max}(\Gamma)\lambda_{\max}(P)}{\lambda_{\min}(P)}} \|\widetilde{v}\|_\infty.$$

The error ε possesses an estimate from Definition 1

$$\|\varepsilon(t)\| \le \beta(\|\varepsilon(0)\|, t) + \gamma(\|d\|_\infty),$$

for any $\varepsilon(0) \in \mathbb{R}^n$ and d, for some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ and the disturbance d is considered to be essentially bounded. Therefore the error e is ISS with respect to $\widetilde{v}$, [20], [21]. Then, the estimation for x and z can be chosen as in (21). □

VI. Examples

Example 1. *Mass-spring system.* To demonstrate a rather practical application, consider a simple mechanical system of two masses connected vertically by a spring with nonlinear stiffness and a linear damper. The bottom mass is also attached to a fixed platform via spring with linear stiffness and a linear damper, and some perturbation u acts at bottom mass (Fig. 1). Then, the system dynamics will be as follows:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -k_1(x_1 - x_3) - k_2(x_1 - x_3)^3 - a_1(x_2 - x_4), \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = k_1(x_1 - x_3) + k_2(x_1 - x_3)^3 + a_1(x_2 - x_4) - k_3x_3 - a_2x_4 + u, \\ y = x_3 + v, \end{cases}$$

where x_1, x_2 are the position and velocity of a top mass, x_3, x_4 are the position and velocity of a second one, k_1, k_2, k_3 are the spring stiffness coefficients, a_1, a_2 are damper coefficients and an input u is some external force or disturbance, acting on a second mass.

Similar simplified representation can be applied, for example, to active suspension models [9], where the top mass represents a quarter-car, and the bottom mass represents a tire, moving on some nonlinear (perturbed) road.

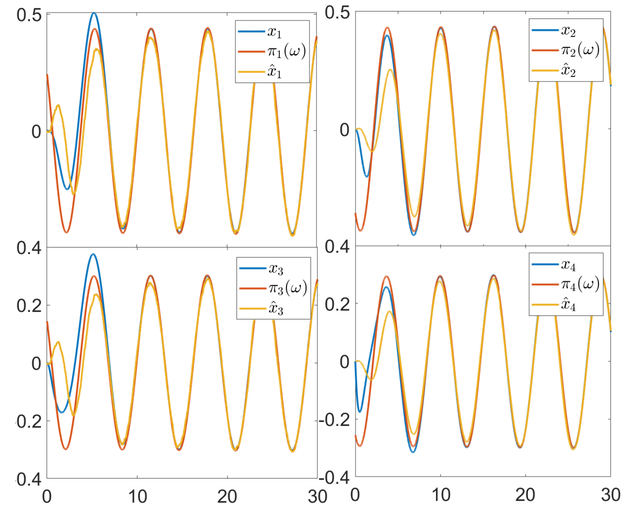

Fig. 2. Simulation results for Example 1. Initial conditions: $x_i(0) = 0$; $\omega_1(0) = 0.1$, $\omega_2(0) = 0$, $\hat{\omega} = 0$. Coefficients: $k_1 = k_2 = 3$, $k_3 = 0.6$, $a_1 = 0.6$, $a_2 = 2$.

The task 2a). Let us assume we want our bottom mass (tire) to follow a certain reference trajectory, produced by a simple signal generator and we want to obtain a representation of our dynamics in terms of generator states ω_1, ω_2. This can be done by finding the proper input $u(\omega)$, using Propositions 1 and 2.

However, the given system is not in the required interlaced form, and we need to do the substitution $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1 - x_3$, $y_4 = x_2 - x_4$. Then, we have following dynamics:

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -k_1y_3 - k_2y_3^3 - a_1y_4, \\ \dot{y}_3 = y_4, \\ \dot{y}_4 = +k_1y_3 + k_2y_3^3 + a_1y_4 - k_3(y_1 - y_3) - a_2(y_2 - y_4) + u. \end{cases}$$

Now we can clearly see that the top subsystem is in upper-triangular form, which allows us to find $y_1(\omega)$ and $y_2(\omega)$ using Proposition 2, having only $y_3(\omega)$. Next logical step is to find $u(\omega)$ by a feedback-like procedure, using the Proposition 1.

Therefore, let us assign $y_3(\omega) := \omega_1 + \omega_2$, then $y_4(\omega) = -\omega_1 + \omega_2$. Let us substitute them in the equation for y_2:

$$\dot{y}_2 = -k_1(\omega_1 + \omega_2) - k_2(\omega_1 + \omega_2)^3 - a_1(-\omega_1 + \omega_2).$$

Expressions for $y_1(\omega)$ and $y_2(\omega)$ can be found by integrating the linear part $\dot{y}_2^{\text{lin}} = \ddot{y}_1^{\text{lin}} = -k_1(\omega_1 + \omega_2) - a_1(-\omega_1 + \omega_2)$, and for the nonlinear part by finding the coefficients of cubic function $y_i(\omega) = y_i^{\text{lin}} + p_{i1}\omega_1^3 + p_{i2}\omega_1^2\omega_2 + p_{i3}\omega_1\omega_2^2 + p_{i4}\omega_2^3$, $i = 1,2$. For our case, the dynamics are as follows:

$$y_1 = y_1^{\text{lin}} - k_2\left(\frac{4}{3}(\omega_2\omega_1^2 - \omega_1\omega_2^2) - \frac{11}{9}(\omega_1^3 + \omega_2^3)\right) := y_1^{\text{lin}} - k_2\tau_1,$$

$$y_2 = y_2^{\text{lin}} + k_2\left(\frac{4}{3}(\omega_1^3 + \omega_2^3) + (\omega_2\omega_1^2 + \omega_1\omega_2^2)\right) := y_2^{\text{lin}} + k_2\tau_2.$$

Next, we substitute $y_1(\omega)$, $y_2(\omega)$, $y_3(\omega)$ in the equation for

y_4, and $u(\omega)$ can be simply found as:

$$u(\omega) = (\omega_1+\omega_2)\left(1-k_1(2-k_3)-a_1a_2-k_3\right)-2k_2(\omega_1+\omega_2)^3 \\ +(\omega_1-\omega_2)\left(a_1(2-k_3)+a_2(1-k_1)\right)-k_2k_3\tau_2+k_2a_2\tau_1.$$

The final step to find $\pi(\omega)$ is to go back to substitution, we have a steady-state solution $\pi(\omega)$ as follows:

$$\begin{cases} \pi_1 = x_1(\omega) = \phi_1 \\ \pi_2 = x_2(\omega) = \phi_2 \\ \pi_3 = x_3 = y_1(\omega) - y_3(\omega) = y_1^{\text{lin}} - k_2\tau_1 - (\omega_1+\omega_2) \\ \pi_4 = x_4 = y_2(\omega) - y_4(\omega) = y_2^{\text{lin}} + k_2\tau_2 - (\omega_2-\omega_1) \end{cases}$$

Fig. 2 shows that the dynamics of x under chosen excitation input u indeed converges to a steady-state solution $\pi(\omega)$.

The task 2b). Let us assume that the system receives some reference signal u, which we do not measure and the system dynamics follows it. Instead, we measure the output of the initial system affected by some white measurement noise: $Y = (x_1 - x_3) + v$. Because of damping, our mechanical system is asymptotically stable and the signal generator oscillates, by design.

The observer would take the form of (21), with the matrices A and C as follows:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

And we can find such L, P, Q and Γ that $A - LC$ is Hurwitz and the LMIs in the theorem 2 are satisfied, for instance for

$$L = \begin{pmatrix} 0.33 & 0.6 \end{pmatrix}^\top.$$

Fig. 2 shows the convergence of an observer to the given steady-state solution of the system.

VII. Conclusion

In this paper, two particular forms of interconnected systems were investigated. The analytical expression for the steady-state solution in a disturbance-free scenario was found. The conditions of existence for such solutions were provided, the result is applied for a robust reduced-order observer design for an interlaced system. The main advantage of the suggested approach is the order reduction of the observer. An example was provided in the last section of the paper.

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