Stabilizing non-discerning control design for discrete-time linear switched systems

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***Abstract*— The distinguishability between the modes of a switched system is an essential concept, particularly for the synthesis of observers. In this paper, we propose to use its opposite, specifically to protect against cyber-physical attacks that aim to obtain information about the system. More precisely, we focus on synthesizing non-discerning control laws that stabilize the switched system. We establish the conditions for the existence of such a law and propose a procedure for its design. The academic examples demonstrate the described application.**

I. Introduction

Switched systems are a long-adopted and actively developing mathematical framework with a solid mathematical background (see [1], [2]).

Linear Time-Invariant switched systems are also sometimes used for the description of Cyber-Physical Systems (CPS), the cyber-security of which is a more recent subject, brought by the rapid development of technology. Almost all of the modern infrastructure and processes, which can be described using switched (or hybrid) systems, contain a layer of communications, which is proven to be corruptible and accessible remotely by the adversary.

The first stage of most cyber-physical attacks is the so-called confidentiality attack, which aims to get data about the system by eavesdropping (gaining access to communications and commands). In the case of switched systems, the problem comes down to the identification of multiple subsystems and modes, depending on the prior knowledge and access to the system by the adversary, which can be challenging.

Despite the associated difficulties due to switching, many mode detection methods (listed, for example, in a survey [3]) deal with the problem efficiently, mainly requiring access to the input/output measurements of the system, with the conditions on mode discernibility (or distinguishability) always being a key aspect. To this end, this paper considers the problem of restricting the ability of the attacker to decipher the mode uniquely. However, the question arises: Can the regular objective of the system still be satisfied for this case (for instance, a stabilization of the state)? What conditions must be imposed on the system for both objectives to align?

A. Related work

With the switching signal allowing a new axis of controllability, the problem of stabilizability changes significantly. The subsystems that are not stabilizable by feedback control can sometimes be stabilized by switching. A summary of earlier results for stabilizability can be found in [4]. For the case of autonomous switched systems, the problem of stabilizability using a switching signal was considered in [5], where new necessary and sufficient conditions based on the set-theory approach were defined. In [6], the definition of periodic stabilizability and the condition in relation to classic stabilizability were extended. In [7], an algorithm using the joint spectral radius to find the stabilizing switching signal was provided. In [8], a new switched Lyapunov function was introduced, based on the results of uncertain systems.

For the case of controlled switched systems, a combination of feedback control and switching signal is used (forming a stabilizing control policy). In [9], a periodic stabilizability was defined, and conditions in relation to classic stabilizability were established. In [10], the existence of piece-wise quadratic control-Lyapunov function and the existence of a stationary exponentially stabilizing hybrid-control policy was proven. In [11], necessary and sufficient conditions for periodic stabilizability and computational tools for upper and lower estimation were provided.

A new problem, introduced by switched systems, is mode discernibility (or sometimes distinguishability [12]), *i.e.* the ability to distinguish between two systems under different switching signals and initiated from different initial conditions, based on the output. Moreover, mode discernibility remains the key aspect for the design and analysis of switched systems; it is a necessary assumption for the state estimation, switched system identification, stabilization, and other problems. Multiple works demonstrated that this concept is closely related to switched systems observability (see [13],[14]), and is often referred to as a mode-observability of the system [15]. The concept is also related to Input Redundancy (IR) [16], if we consider the control and the switching signal as inputs for the system.

Control design plays a crucial role in mode distinguishability. The so-called discernible control was introduced in [15], and later extended for the case of external disturbances in [17]. The so-called indistinguishable zone for a class of initial vectors and input signals was defined in [18]. The results were also applied to the cybersecurity scenario in [19], where the goal of the CPS operator was to keep the modes discernible at all times regarding possible corruption of the input by the adversary, in [20] the stealthy Man-in-the-Middle attack was later designed using the corrupted discerning input.

To the best of the authors' knowledge, the non-discerning control appeared only a few times in the literature and was never considered the design objective. For example, in [17] and [21], the set of non-discerning control sequences was defined for discrete-time linear systems, which led the state to an indistinguishability zone, a scenario to be avoided.

Following the motivation and gaps presented above, the paper offers the following contributions:

- novel conditions for the existence of a non-discerning control for switched systems;
- conditions for stabilizability under the non-discerning control;
- a constructive design procedure satisfying both objectives.

B. Outline of the paper

The paper is structured as follows: Section II provides with the notation, a system definition, stabilizability and mode discernibility notions, which are used to express the main result, presented in Section III. Section IV offers a detailed control design, based on the results expressed in the previous section, Section V demonstrates the application of the result to a CPS under external observation; Section VI is dedicated to academic examples, which validate the approach; and Section VII concludes the paper with remarks and future directions of the work.

II. Preliminaries

A. Notation

- The set of real numbers is denoted by $\mathbb{R}$, while the set of natural numbers by $\mathbb{N}$, the spaces of real vectors of dimension n_x and real matrices of dimension $n_x \times m_x$ are denoted by $\mathbb{R}^{n_x}$ and $\mathbb{R}^{n_x \times m_x}$, respectively.
- $\|\cdot\|$ refers to the Euclidian norm in $\mathbb{R}^{n_x}$, while rank$(\cdot)$, Im$(\cdot)$, dim$(\cdot)$ refer to a rank, image and dimension of a matrix, correspondingly.
- Let $\lambda(\cdot)$ be the eigenvalue of the matrix, $\Omega(\cdot)$ be the spectrum of the matrix. Denote by I_{n_x} the $n_x \times n_x$ an identity matrix, and with 0_{n_x} the $n_x \times n_x$ zero matrix. Let the $(\cdot)^\dagger$ refer to the Moore–Penrose inverse of the matrix.

B. System definition

Consider a switched linear time-invariant system in discrete-time:

$$\begin{cases} x(k+1) &= A_i x(k) + B_i u(k), \\ y(k+1) &= C_i x(k+1), \end{cases} \tag{1}$$

with $x(k) \in \mathbb{R}^{n_x}$ as the state, $u(k) \in \mathbb{R}^{n_u}$ as the input, $y(k) \in \mathbb{R}^{n_y}$ as the measured output at time instant k. Denote $Q := 1, \ldots, q$ $(q \in \mathbb{N})$, then $A_i, B_i, C_i, i \in Q$ are matrices of appropriate dimensions, and $\sigma : \mathbb{N} \to Q$, is the switching signal. For the clarity of exposition, let the notation $\sigma(k)$ and σ be interchangeable for the rest of the paper. Define $u_t = \{u(k)\}_{k=0}^{t-1}$ and $\sigma_t = \{\sigma(k)\}_{k=0}^{t-1}$ as sets of inputs and modes up to the time instant t, respectively; trajectories starting from an initial state x_0 as $x(t, x_0, u_t, \sigma_t)$ and $y(t, x_0, u_t, \sigma_t)$; and π_t as the sequence of the pairs of discrete inputs $\pi_t := (u_k, \sigma_k)_{k=0}^{t-1}$. The switching law is said to be n-periodic if $\forall k \in \mathbb{N}, \sigma(k+n) = \sigma(k)$.

Remark 1. *Note that with the system defined as in* (1)*, the mode $\sigma(k) = i$ is defined in a way that the output $y(k+1)$ is related to the dynamics of $x(k+1)$, which differs from the conventional definition where C_i corresponds to $y(k)$. The idea behind this is to simplify the notation for indices and to highlight the use of the expected output $y(k+1)$ for the design of the input $u(k)$.*

Let $\mathscr{F} = \{A_i\}_{i=1}^q$ be the set of matrices of the switched linear system (1). Then, for some $n \in \mathbb{N}$, denote by $\Sigma_n(\mathscr{F})$ the set of length-n products of $\mathscr{F}$

$$\Sigma_n(\mathscr{F}) = \left\{ \prod_{j=0}^{n-1} A_{i_j}, i_j \in \{1, \ldots, q\} \right\}. \tag{2}$$

C. Stabilizability

Definition 1. *[11] The system* (1) *is called exponentially stabilizable with parameters $a \geq 0$ and $c \in [0, 1)$ if starting from any initial state $x_0 \in \mathbb{R}^{n_x}$, there exists a sequence π_t such that for all $t \in \mathbb{N}$*

$$\|x(t, x_0, \pi_t)\| \leq a c^t \|x_0\|. \tag{3}$$

The system (1) is called periodic stabilizable if the switching law in π_t is periodic.

D. Mode discernibility

Consider a copy of the system (1), denote its initial state, state, switching signal, and output as $\bar{x}_0$, $\bar{x}(k)$, $\bar{\sigma}$, $\bar{y}(k)$, respectively. We propose to introduce the notion of non-discerning control, which is defined as follows:

Definition 2. *The control $u(k)$ is called* non-discerning control *for a system* (1) *if for all states $x(k), \bar{x}(k)$ there exist $\sigma, \bar{\sigma}, (\sigma \neq \bar{\sigma})$ such that the corresponding outputs are equal*

$$\forall x(k), \bar{x}(k),\ \exists (\sigma, \bar{\sigma}) : y(k+1, x(k), u(k), \sigma) = \bar{y}(k+1, \bar{x}(k), u(k), \bar{\sigma}). \tag{4}$$

Let two modes $(\sigma, \bar{\sigma})$ satisfying (4) be called controlled-indiscernible.

Let us denote the switching sequence that leads to (3) as σ_t^*. Then, the objective of the paper is to establish conditions on a system (1) under which both (4) and (3) are satisfied. Another objective is to provide with a constructive design for $\pi_t = (u_k, \sigma_k^*)_{k=0}^{t-1}$ when the conditions are satisfied.

III. Main Result

A. Existence of the non-discerning control

Proposition 1. *There exists a non-discerning control for the system* (1)*, if and only if there exists at least one pair $(i, j) \in Q \times Q$, $(i \neq j)$ such that the following rank condition is satisfied*

$$rank\left(\begin{bmatrix} C_i A_i & -C_j A_j & G_{ij} \end{bmatrix}\right) = rank(G_{ij}), \tag{5}$$

where $G_{ij} := (C_j B_j - C_i B_i)$.

Moreover, for any $k \geq 0$ and $\forall x(k), \bar{x}(k)$ the control $u(k)$ can be defined as follows

$$u(k) = G_{ij}^\dagger \left(C_i A_i x(k) - C_j A_j \bar{x}(k)\right). \tag{6}$$

Let us denote with $\bar{Q} \subset Q \times Q$ the set of all pairs $(i,j), i \neq j$, such that (5) is satisfied.

Proof. The rank condition in Proposition 1 is obtained from the resolvability of the equation

$$y(k+1) - \bar{y}(k+1) = C_i A_i x(k) - C_j A_j \bar{x}(k) - G_{ij} u(k). \quad (7)$$

$\Rightarrow$ (if): For the sufficiency of the claim, it is enough to demonstrate that if the rank condition (5) is satisfied, then a solution for $u(k)$ exists (for any $x(k)$, $\bar{x}(k)$) such that (7) equals to 0.

Assume that the condition (5) is satisfied, and recall that $\dim(\mathrm{Im}(A)) = \mathrm{rk}(A)$, it leads to

$$\mathrm{Im}\left(\begin{bmatrix} C_i A_i & -C_j A_j & G_{ij} \end{bmatrix}\right) = \mathrm{Im}(G_{ij}) \quad (8)$$

which leads to $\mathrm{Im}\left(\begin{bmatrix} C_i A_i & -C_j A_j \end{bmatrix}\right) \subseteq \mathrm{Im}(G_{ij})$ which means that for any $x(k)$, $\bar{x}(k)$ there exists an input $u(k)$ such that

$$y(k+1) - \bar{y}(k+1) = C_i A_i x(k) - C_j A_j \bar{x}(k) - G_{ij} u(k) = 0$$

with $u(k) = G_{ij}^{\dagger}(C_i A_i x(k) - C_j A_j \bar{x}(k))$.

$\Leftarrow$ (only if): For the necessity of the claim, we assume that the equation (7) equals 0, and then demonstrate that the equation is not solvable for $u(k)$ (for any $x(k)$,$\bar{x}(k)$) only in the case when the rank condition is not satisfied.

Let us assume the contrary, that there exists non-discerning control for the system (1), and the condition (5) is not satisfied, *i.e.*

$$\mathrm{rank}\left(\begin{bmatrix} C_i A_i & -C_j A_j & G_{ij} \end{bmatrix}\right) > \mathrm{rank}(G_{ij}). \quad (9)$$

We have

$$C_i A_i x(k) - C_j A_j \bar{x}(k) = G_{ij} u(k) \quad (10)$$

Consider the case where $u(k) \in \ker(G_{ij})$, then

$$C_i A_i x(k) = C_j A_j \bar{x}(k), \quad (11)$$

which can only be true for any $x(k)$,$\bar{x}(k)$ if $\mathrm{rank}\left(\begin{bmatrix} C_i A_i & C_j A_j \end{bmatrix}\right) = 0$, which contradicts (9) no matter the rank of G_{ij}.

In order for (10) to be satisfied for any $x(k)$,$\bar{x}(k)$, where $u(k) \notin \ker(G_{ij})$ the following has to hold

$$\mathrm{Im}(\begin{bmatrix} C_i A_i & C_j A_j \end{bmatrix}) \subseteq \mathrm{Im}(G_{ij}) \quad (12)$$

$$\mathrm{rank}(\begin{bmatrix} C_i A_i & C_j A_j \end{bmatrix}) \leq \mathrm{rank}(G_{ij}) \quad (13)$$

which leads to (5) and contradicts (9). □

After getting the expression and conditions for the existence of non-discerning control, the question of the stabilizability of system (1) under u_t can be addressed. It is evident from Proposition 1 that the behavior of both states $x(k)$ and $\bar{x}(k)$ must be considered together.

B. Stabilizability of an augmented system

Let us define a system for the augmented state $X := \begin{pmatrix} x(k)^\top & \bar{x}(k)^\top \end{pmatrix}^\top$ as follows

$$X(k+1) = \begin{pmatrix} A_i & 0 \\ 0 & A_j \end{pmatrix} X(k) + \begin{pmatrix} B_i \\ B_j \end{pmatrix} u(k), \quad (14)$$

$$Y(k+1) = \begin{pmatrix} C_i & -C_j \end{pmatrix} X(k+1). \quad (15)$$

By design, two modes σ and $\bar{\sigma}$ are controlled-indiscernible for a switched system (1) if the output of the augmented system (14)-(15) $Y \equiv 0$ for all $k \geq 0$. Let the conditions of the Proposition 1 be satisfied, then under non-discerning control (6) we get an autonomous system with $Y \equiv 0$.

$$X(k+1) = \begin{pmatrix} A_i + B_i G_{ij}^{\dagger} C_i A_i & -B_i G_{ij}^{\dagger} C_j A_j \\ B_j G_{ij}^{\dagger} C_i A_i & A_j - B_j G_{ij}^{\dagger} C_j A_j \end{pmatrix} X(k) := \Phi_{ij} X(k) \quad (16)$$

for all $(i,j) \in \bar{Q}$.

Remark 2. *By construction, the augmented system has the following property:*

$$\Phi_{ij} = \begin{pmatrix} 0_{n_x \times n_x} & I_{n_x \times n_x} \\ I_{n_x \times n_x} & 0_{n_x \times n_x} \end{pmatrix} \Phi_{ji} \begin{pmatrix} 0_{n_x \times n_x} & I_{n_x \times n_x} \\ I_{n_x \times n_x} & 0_{n_x \times n_x} \end{pmatrix} \quad (17)$$

which leads to the same spectrum for both matrices $\Omega(\Phi_{ij}) \equiv \Omega(\Phi_{ji})$, for all $(i,j) \in \bar{Q}$.

Denote the index $p := ij$, and consider the collection of matrices $\bar{\mathscr{F}} = \{\Phi_p\}_{p=1}^{m}$, where m is the cardinality of the set $\bar{Q}$. For each timestep $n \in \mathbb{N}$, we define the set $\Sigma_n(\bar{\mathscr{F}})$ containing all possible products of degree n:

$$\Sigma_n(\bar{\mathscr{F}}) = \left\{ \prod_{i=0}^{n-1} \Phi_{p_i} : p_i \in \{1, \ldots, m\} \right\} \quad . \quad (18)$$

Theorem 1. *If the augmented system* (16) *is exponentially stabilizable, then the system* (1) *is exponentially stabilizable under non-discerning control u_t.*

Proof. $\Rightarrow$: Since $\|X_t\| \leq \bar{a}\bar{c}^t \|X_0\|$, for any X_0 there exists a switching sequence $\sigma^* = (\sigma_t^*, \bar{\sigma}_t^*)$ such that

$$\|X(t, X_0, \sigma^*)\| \leq \bar{a}\bar{c}^t \|X_0\|. \quad (19)$$

Without loss of generality, let us assume $t = 1$ ($\sigma_t^* = \{i\}$, $\bar{\sigma}_t^* = \{j\}$). Then we obtain

$$X_t = \begin{pmatrix} A_i + B_i G_{ij}^{\dagger} C_i A_i & -B_i G_{ij}^{\dagger} C_j A_j \\ B_j G_{ij}^{\dagger} C_i A_i & A_j - B_j G_{ij}^{\dagger} C_j A_j \end{pmatrix} X_0 = \begin{pmatrix} A_i & 0 \\ 0 & A_j \end{pmatrix} X_0 + \begin{pmatrix} B_i G_{ij}^{\dagger} C_i A_i & -B_i G_{ij}^{\dagger} C_j A_j \\ B_j G_{ij}^{\dagger} C_j A_j & -B_j G_{ij}^{\dagger} C_i A_i \end{pmatrix} X_0. \quad (20)$$

The state of the system (1) can be expressed as follows

$$x_t = A_i x_0 + B_i G_{ij}^{\dagger}(C_i A_i x_0 - C_j A_j \bar{x}_0) = A_i x_0 + B_i u_t. \quad (21)$$

We obtain a control policy $\pi_t = (u_t, \sigma_t^*)$, under which starting from any x_0, the system (1) is stabilizable (with $a = \bar{a}\|\bar{x}_0\|, c = \bar{c}$):

$$\|x(t, x_0, \pi_t)\| \leq \|X(t, X_0, \sigma^*)\| \leq \bar{a}\bar{c}^t \|X_0\| \leq \bar{a}\bar{c}^t \|x_0\| \|\bar{x}_0\| \leq ac^t \|x_0\|.$$

Moreover, the input u_t is non-discerning, *i.e.* for $(\sigma_t^*, \bar{\sigma}_t^*)$ and $\forall x_0, \bar{x}_0 : y(t, x_t, u_t, \sigma_t^*) = \bar{y}(t, \bar{x}_t, u_t, \bar{\sigma}_t^*)$, due to the design of the system (16). □

Remark 3. *Generally speaking, the necessity does not follow: there exist counterexamples when rank condition is satisfied, the system (1) is stabilizable for any x_0 under some policy $\pi_t = (\sigma_t, u_t)$ and for some $\bar{x}_0, \bar{\sigma}_t$, but there does not exist a switching sequence $\sigma^* = (\sigma_t^*, \bar{\sigma}_t^*)$ that an autonomous system built using the non-discerning control is stabilizable. Example 2 in Section VI demonstrates the idea.*

The following section addresses a practical approach for determining the periodic stabilizing switching sequence σ^*, based on the joint spectral subradius [22] for the augmented system (16). If such a sequence is found, the design of the non-discerning control follows, and the complete control policy π_t is ready to be applied to the system (1).

IV. Numerical implementation

A. Reduced set size determination

Recall that the spectral radius of a matrix A_σ is defined as follows

$$\rho(A_\sigma) = \max\{|\lambda_1(A_\sigma)|, \ldots, |\lambda_{n_x}(A_\sigma)|\} = \lim_{n\to\infty} \|A_\sigma^n\|^{\frac{1}{n}}. \quad (22)$$

And the so-called *joint spectral subradius* [22] as

$$\check{\rho}(\mathscr{F}) = \liminf_{n\to\infty}\{\rho(A_\sigma)^{1/n} : A_\sigma \in \Sigma_n(\mathscr{F})\}, \quad (23)$$

where $\mathscr{F} = \{A_1, \ldots, A_q\}$, $\Sigma_n(\mathscr{F}) := \left\{\prod_{j=0}^{n-1} A_{i_j} : i_j \in Q\right\}$.

The joint spectral subradius calculation can be a computationally challenging task. Therefore, in order to optimize the design, the set $\Sigma_n(\mathscr{F})$ can be efficiently reduced to the set $S_n \subset \Sigma_n(\mathscr{F})$ using the properties of the spectral radius.

For example, recall the known properties of the cyclic repetition

$$\rho(A_1 \cdots A_\theta A_{\theta+1} \cdots A_\tau) \equiv \rho(A_{\theta+1} \cdots A_\tau A_1 \cdots A_\theta). \quad (24)$$

And of the equivalency of matrix products

$$\rho(A_1 \cdots A_\theta)^\tau > \rho(A_1 \cdots A_\theta) \quad \text{if} \quad \rho(A_1 \cdots A_\theta) > 1, \quad (25)$$
$$\rho(A_1 \cdots A_\theta)^\tau < \rho(A_1 \cdots A_\theta) \quad \text{if} \quad \rho(A_1 \cdots A_\theta) < 1. \quad (26)$$

The number of unique combinations (due to the property (24)) can be obtained using a classical result from combinatorics based on the MacMahon Theorem.

Lemma 1. *[Theorem 1 in [23]] The number of cyclically different combinations of length n from a set of m characters is*

$$M(n,m) = \frac{1}{n}\sum_{d\backslash n} m^d \phi\left(\frac{n}{d}\right), \quad (27)$$

where ϕ is the Euler totient function. ($\phi(l)$ is the number of the integers if $\{0, \ldots, l-1\}$ which are relatively prime to l. The notation $d\backslash n$ indicates (summation over) positive divisors, that is, $n = kd$ where k is an integer and d is a positive integer.)

Furthermore (from [23]), due to (25)-(26) we have

$$N(n,m) = \sum_{d\backslash n} \mu(d) M\left(\frac{n}{d}, m\right), \quad (28)$$

where μ is the Mobius function.

In addition to a known size reduction described above, according to Remark 2, in our case the following is satisfied

$$\rho(\Phi_{ij}) \equiv \rho(\Phi_{ji}). \quad (29)$$

This leads to the significant size reduction on top of (28), for the case of a system under non-discerning control: the cardinality of the set S_n can be calculated as follows

$$\bar{N}(n,m) = \frac{N(n,m) + \frac{m(1+(-1)^n)}{4}}{2}. \quad (30)$$

Given a number of timesteps $n \in \mathbb{N}$ and using all the properties defined above, the following algorithm can be designed to obtain a set S_n.

Algorithm 1: Set reduction

Build a $m^n \times n$ matrix of indexes (denoted as $\bar{\Sigma}_n$) for all combinations of $\Sigma_n(\mathscr{F})$, determine $\bar{N}(n,m)$ using equation (30).

for $i = 1 : \bar{N}(n,m)$ **do**

Take a row from $\bar{\Sigma}_n(i,:)$ (denoting it as $p_n, \ldots, p_1$), and extend it to the right as $\{p_n, \ldots, p_1, p_n, \ldots, p_1\}$.

1. Due to (29), we can delete its pair row $\{p_n + (-1)^{p_n+1}, \ldots, p_1 + (-1)^{p_1+1}\}$ from the matrix $\bar{\Sigma}_n$.

2. Due to the (25)-(26), if the combination $\{p_n, \ldots, p_1\}$ appears only twice in the extended sequence consecutively, proceed to Step 3. If not, exclude the row.

3. Due to (24), the spectral radius of all consecutive combinations of length n in $\{p_n, \ldots, p_1, p_n, \ldots, p_1\}$ is equal. Therefore, find and remove all the rows appearing as a combination from the matrix $\bar{\Sigma}_n$, except for $\{p_n, \ldots, p_1\}$ itself.

end for

4. At the end of the algorithm, the matrix of indexes $\bar{\Sigma}_n$ is efficiently reduced to the cardinality of the set S_n (*i.e* to $\bar{N}(n,m)$).

B. Determination of stabilizability and control design

Recall the well-known result on the periodic stabilizability of autonomous systems.

Lemma 2 (Lemma 4 in [6])**.** *The system (16) is periodic stabilizable if and only if there exists a sequence of matrices $\mathbb{A}_n^* := \prod_{j=0}^{n-1} \Phi_{p_j} = \Phi_{p_{n-1}} \cdots \Phi_{p_0}$ which is Schur.*

Then the goal is then to find (for the smallest n) $\mathbb{A}_n{}^*$ which is Schur for the augmented system (16). Using the set reduction technique, the following *branch-and-bound* algorithm for the calculation of $\check{\rho}(\bar{\mathscr{F}})$ and determination of σ_n^* can be designed.

Algorithm 2: Determination of stabilizabilty of (16).

1\. Let $\check{\rho}(\bar{\mathscr{F}}) := +\infty$ **and** $n := 1$.
while $\check{\rho}(\bar{\mathscr{F}}) > 1$ **and** $n \leq n_{max}$ **do**
  2. Determine the set of all combinations $\Sigma_n(\bar{\mathscr{F}})$.
  3. Determine the set $S_n \in \Sigma_n(\bar{\mathscr{F}})$ using Algorithm 1.
  4. Compute all spectral radii for S_n,
  if $\exists \mathbb{A}_n \in S_n$ such that $\rho(\mathbb{A}_n) < 1$ **then**
    $\check{\rho}(\bar{\mathscr{F}}) = \rho(\mathbb{A}_n)$, $\mathbb{A}_n^* := \mathbb{A}_n$.
  end if
  $n = n+1$.
end while

If the algorithm does not find a stabilizing switching sequence, we state that the augmented system (16) is not stabilizable in n_{max} steps. If $\mathbb{A}_n^* = \prod_{k=0}^{n-1} \Phi_{i_k j_k} = \Phi_{i_{n-1} j_{n-1}} \cdots \Phi_{i_0 j_0}$ was found, we can extract stabilizable switching signals $\sigma_n^* = \{i_k\}_{k=0}^{n-1}$ and $\bar{\sigma}_n^* = \{j_k\}_{k=0}^{n-1}$ and proceed with the design of the non-discernible control u_n to complete the control policy $\pi_n = (u_k, \sigma_k)_{k=0}^{n-1}$. Applying the equation (6) for the interval $(0, n-1)$, for a given x_0 we can pick any $\bar{x}_0$ for the copy of the system, and we get the following:

$$u(k) = G_{i_k j_k}^{\dagger}\left(C_{i_k} A_{i_k} x(k) - C_{j_k} A_{j_k} \bar{x}(k)\right), \; \forall k = 0, \ldots, n-1. \tag{31}$$

The obtained control policy π_n can be repeated periodically until the stabilization of the state x is achieved.

V. Application

Consider a a window of measurements $[t-\bar{t}, t]$ of switched system (1) controlled by the non-discerning control under external observation.

$$y_{[t-\bar{t},t]} = \mathscr{O}_{[t-\bar{t},t]} x_0 + \mathscr{H}_{[t-\bar{t},t]} U_{[t-\bar{t},t]} \tag{32}$$

where $U_{[t-\bar{t},t]}$ and $y_{[t-\bar{t},t],r}$ are measured input and output during a chosen time window, and

$$\mathscr{O}_{[t-\bar{t},t]} = \begin{pmatrix} C_i \\ C_i A_i \\ \cdots \\ C_i A_i^{t-1} \end{pmatrix}, \; \mathscr{H}_{[t-\bar{t},t]} = \begin{pmatrix} 0 & & & \\ C_i B_i & \ddots & & \\ \vdots & \ddots & \ddots & \\ C_i A_i^{t-2} B_i & \cdots & C_i B_i & 0 \end{pmatrix}.$$

Let us apply the method of active path determination from [24]. To exclude the state in the input-output monitor, the vector w is considered, which satisfies the following

$$w^{\top} \mathscr{O}_{[t-\bar{t},t]} = 0. \tag{33}$$

Then, the residual $r_{[t-\bar{t},t]}$ is introduced which can be defined as follows:

$$r_{[t-\bar{t},t]} = w^{\top} y_{[t-\bar{t},t]} - w^{\top} \mathscr{H}_{[t-\bar{t},t]} U_{[t-\bar{t},t]}. \tag{34}$$

In order to deduce the active path from all the possible options, all the residuals have to be checked. It is evident that the residual $r_{[t-\bar{t},t]}(i)$ will correspond to 0 for σ_t. However, since $y_{[t-\bar{t},t]} \equiv \bar{y}_{[t-\bar{t},t]}$ by definition of the non-discerning control $U_{[t-\bar{t},t]}$, the residual $r_{[t-\bar{t},t]}(j)$ in the case of the sequence $\bar{\sigma}_t$ will also correspond to zero. Therefore, the active mode cannot be determined uniquely using the active path determination, which in turn implies that the state cannot be determined uniquely.

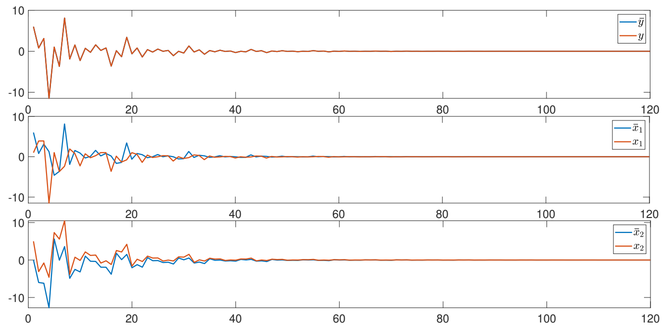

Fig. 1: Non-discerning control for Example 1, initialized from $x_0 = (1 \quad 5)^{\top}$, $\bar{x}_0 = (6 \quad 0)^{\top}$

VI. Examples

A. Example 1

Consider the switched system (1) with 2 modes as follows

$$A_1 = \begin{pmatrix} 2 & 0 \\ -1 & 0.9 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1.4 & 0.5 \\ 0 & 0.5 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

Since $\operatorname{rank}\left(\begin{bmatrix} C_1 A_1 & -C_2 A_2 & G_{12} \end{bmatrix}\right) = \operatorname{rank}(G_{12}) = 1$, the non-discerning control can be designed. We obtain a set $\bar{\mathscr{F}}$ of two matrices Φ_{12} and Φ_{21} as follows

$$\Phi_{12} = \begin{pmatrix} -2 & 0 & 2.8 & 2 \\ -1 & 0.9 & 0 & 0 \\ 0 & 0 & 1.4 & 0.5 \\ -2 & 0 & 1.4 & 1.5 \end{pmatrix}, \; \Phi_{21} = \begin{pmatrix} 1.4 & 0.5 & 0 & 0 \\ 1.4 & 1.5 & -2 & 0 \\ 2.8 & 2 & -2 & 0 \\ 0 & 0 & -1 & 0.9 \end{pmatrix}.$$

Let us apply Algorithm 2:
2) In case of $n = 1$, $\Sigma_1(\bar{\mathscr{F}}) = \bar{\mathscr{F}} = \{\Phi_{12}, \Phi_{21}\}$;
3) We exclude one of the matrices due to the inverse combination (*i.e.* $\rho(\Phi_{12}) = \rho(\Phi_{21})$), and $S_1 = \{\Phi_{12}\}$;
4) We obtain $\rho(\Phi_{12}) = 4.67$, which forces us to increase n; Repeating steps 2)-4) until $n = 6$, we do not succeed, the first stable combination can be found only as $\rho(\Phi_{12}^3 \Phi_{21}^3) = 0.62$, which concludes that the system is periodic stabilizable. Using the equation (31), we design the control policy which stabilizes the state in $\sim$ 70 timesteps. (see Figure 1).

Let us now apply the active path determination for the case of non-discerning control. Without loss of generality, choose a short time window $\bar{t} = 2$, $t = 60$, which corresponds to a mode sequence $\sigma_{[t-\bar{t},t]} = \{1,1,1\}$ (due to the Algorithm 2). Using the matrices for both modes, let us design $w(i)$ using (33): as expected, at least two residuals corresponding to the sequences $\sigma_{[t-\bar{t},t]}$ and $\bar{\sigma}_{[t-\bar{t},t]} = \{2,2,2\}$ are zero, corresponding to $w(1) = \begin{pmatrix} -6 & 1 & 1 \end{pmatrix}$ and $w(2) = \begin{pmatrix} 0.7 & -1.9 & 1 \end{pmatrix}$.

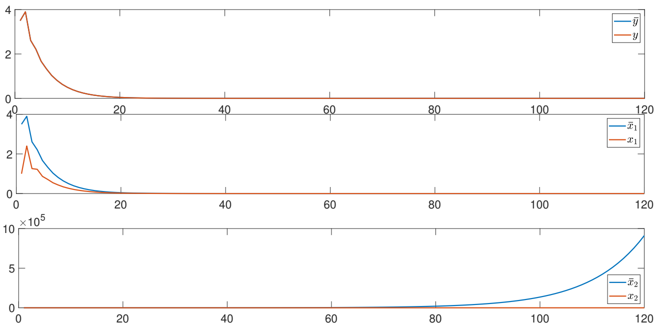

Fig. 2: Example 2. Counterexample for the necessity of Theorem 1, initialized from $x_0 = \begin{pmatrix}1 & 5\end{pmatrix}^\top$, $\bar{x}_0 = \begin{pmatrix}6 & 0\end{pmatrix}^\top$

B. Example 2. Counterexample for the Theorem 1

Consider the switched system as follows

$$A_1 = \begin{pmatrix} 0 & -0.8 \\ 0.8 & 1.1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.4 \\ 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} -0.1 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 0.5 \end{pmatrix}.$$

The rank condition is satisfied, therefore, we can design the augmented system (16). The search for the combination concludes in not stabilizability (for $n_{max} = 25$).

However, looking at eigenvalues $\lambda(\Phi_{ij}) = \{1.1, 0.78, -0.38, 0\}$ we see that only one subsystem is unstable (corresponding to mode 1), and the second one is stable, but not controllable (due to $B_2 = \begin{pmatrix}0 & 0\end{pmatrix}^\top$). As a result, choosing, for example, the switching signals as $\sigma_t = \{2, 2, \ldots, 2\}$ and $\bar{\sigma}_t = \{1, 1, \ldots, 1\}$, we obtain the non-discerning control u_t which stabilizes the system (1), while the state of the $\bar{x}$ goes to infinity (see Figure 2).

VII. Conclusion and Future Work

The results presented in this paper introduce the concept of non-discerning stabilizing control, the objective of which is to keep the switched system in a non-distinguishable state while bringing the state of the system to zero. The conditions for the existence of such a control appeared to be easily determined using a rank condition on a system. The stabilizability condition was ensured, based on the augmented system, assembled using the two modes under non-discernible control. The design procedure presented in Section IV uses the fact that for the stabilizability of the augmented system, it is enough to find a joint spectral subradius, less than one, while reducing the computational cost of such a process by excluding most combinations. The algorithm is constructive and ready to be applied to the system, for example, under a confidentiality attack (as demonstrated in Section V). The academic example validates the approach by proving the inability of the attacker to determine the mode and state uniquely.

Future works include consideration of disturbances and measurement noise, which significantly changes the problem statement and requires an updated design and rigorous analysis of the problem. Extension of the result to other types of systems (nonlinear, LTV, etc.) is also a relevant direction.

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